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On the Figure and Stability of a Liquid Satellite

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IV. *On the Figure and Stability of a Liquid Satellite.*

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Preface.

More than half a century ago Édouard Roche wrote his celebrated paper on the form assumed by a liquid satellite when revolving, without relative motion, about a solid planet.* In consequence of the singular modesty of Roche's style, and also because the publication was made at Montpellier, this paper seems to have remained almost unnoticed for many years, but it has ultimately attained its due position as a classical memoir.

The laborious computations necessary for obtaining numerical results were carried out, partly at least, by graphical methods. Verification of the calculations, which as far as I know have never been repeated, forms part of the work of the present paper. The distance from a spherical planet which has been called "Roche's limit" is expressed by the number of planetary radii in the radius vector of the nearest possible infinitesimal liquid satellite, of the same density as the planet, revolving so as always to present the same aspect to the planet. Our moon, if it were homogeneous, would have the form of one of Roche's ellipsoids; but its present radius vector is of course far greater than the limit. Roche assigned to the limit in question the numerical value 2·44; in the present paper I show that the true value is 2·455, and the closeness of the agreement with the previously accepted value affords a remarkable testimony to the accuracy with which he must have drawn his figures.

He made no attempt to obtain numerical solutions except in the case of the infinitely small satellite. In this case the figure is rigorously ellipsoidal, but for a finite satellite this is no longer the case; nor do his equations afford the means of determining exactly the ellipsoid which most nearly represents the truth. These deficiencies are made good below, and we find that even in the extreme case of two equal masses in limiting stability the ellipsoid is a much closer approximation to accuracy than might have been expected.

It is natural that Roche, writing as he did half a century ago, should not have been in a position to discuss the stability of his solutions with completeness, and although he did much in that direction he necessarily left a good deal unsettled.

In 1887 I attempted the discussion of some of the problems to which this paper is devoted, by means of spherical harmonic analysis.† Poincaré's celebrated memoir on figures of equilibrium‡ was published just when my work was finished, and I kept my paper back for a year in order to apply to my solutions the principles of stability

* "La figure d'une masse fluide soumise à l'attraction d'un point éloigné," 'Acad. des Sci. de Montpellier,' vol. 1, 1847–50, p. 243.

† "Figures of Equilibrium of Rotating Masses of Fluid," 'Phil. Trans. Roy. Soc.,' vol. 178 (1887) A, pp. 379–428.

‡ "Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation," 'Acta. Math.' 7: 3, 4 (1885), pp. 259–380.

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enounced by him. The attempt is given in an appendix to my paper, but unfortunately I failed to understand his work completely, and my investigation, as it stands, is erroneous from the fact that one term in the energy is omitted.* I think, however, that the defect may easily be made good.

The analysis of the present paper is carried out by means of ellipsoidal harmonic analysis. In the course of the work it becomes necessary to refer to previous papers by myself, all published in the 'Philosophical Transactions'; they are: "Ellipsoidal Harmonic Analysis," vol. 197 (1901) A, pp. 461–557; "The Pear-Shaped Figure of Equilibrium of a Rotating Mass of Liquid," vol. 198 (1902) A, pp. 301–331; "The Stability of the Pear-Shaped Figure of Equilibrium, &c.," vol. 200 (1903) A, pp. 251–314; "The Integrals of the Squares of Ellipsoidal Surface Harmonic Functions," vol. 203 (1904) A, pp. 111–137. These papers are hereafter referred to by the abridged titles "Harmonics," "The Pear-Shaped Figure," "Stability," and "Integrals."

The analysis involved in the investigation is unfortunately long and complicated, but the subject itself is not an easy one, and the complication was perhaps unavoidable.

The principal inducement to attack this problem was the hope that it might throw further light on the form of the pear-shaped figure in an advanced stage of development when it might be supposed to consist of two bulbs of liquid joined by a very thin neck. The arguments adduced below seem to show that such a figure must be unstable.

M. LIAPOUNOFF has recently published a paper in which he states that he is able to prove the instability of the pear-shaped figure even when only infinitesimally furrowed.† In view of my previous work on the stability of this figure, and from other considerations it seems very difficult to accept the correctness of this result.

At the end a summary is given of the conclusions arrived at, and this last subject is discussed amongst others.

PART I.—ANALYSIS.

§ 1. *The Stability of Liquid Satellites.*

This paper deals with two problems concerning liquid satellites which possess so much resemblance that I did not for some time perceive that there is an essential difference between them. One of these is the determination of the figures and of the secular stability of two masses of liquid revolving about one another in a circular orbit without relative motion of their parts. We may refer to this as the problem of

* This is the term denoted $\frac{1}{2}\omega^2(\delta I)^2/I$ below.

† "Sur un problème de Tchebychef," 'Acad. Imp. des Sci. de St. Pétersbourg,' vol. 17, No. 3 (1905).

"the figures of equilibrium"; the other may be called "ROCHE's problem," and it differs only from the former in that one of the two masses of liquid is replaced by a particle or by a rigid sphere. However, in the numerical solutions found hereafter, ROCHE's problem is slightly modified, for the rigid sphere is replaced by a rigid ellipsoid of exactly the same form as that assumed by the other mass of liquid in the problem of the figures of equilibrium. Thus, with this modification, the two problems become identical as regards the shape of the figures; but, as we shall see, they differ widely as to the conditions of secular stability. This difference arises from the fact that in the one case there are two bodies which may be subject to tidal friction, and in the other there is only one.

If in either problem there is no solution when the angular momentum has less than a certain critical value, if for that value there is one solution and for greater values there are two, then the principle of POINCARÉ shows that the single solution is the starting point of a pair of which one has one fewer degrees of instability than the other. If, then, one of the two solutions is continuous with a solution which is clearly stable, it follows that the determination of minimum angular momentum will give us the limiting stability of that solution; and this is the point of greatest interest in all such problems.

Our two problems differ in the value of the angular momentum of which the minimum has to be found. For, if in ROCHE's problem the second body is a particle, it has only orbital momentum; if the second body is a sphere, it must be deemed to have no rotation; and, finally, in the modified form of the problem, the rotational momentum of the rigid body must be omitted from the angular momentum, which has to be a minimum for limiting stability.

It will be useful to make a rough preliminary investigation of the regions in which we shall have to look for cases of limiting stability in the two problems. For this purpose I consider the case of two spheres as the analogue of the problem of the figure of equilibrium, and the case of a sphere and a particle as the analogue of ROCHE's problem.

Let ρ be density, and let the mass of the whole system be $\frac{4}{3}\pi\rho a^3$; let the masses of the two spheres be $\frac{4}{3}\pi\rho a^3\lambda/(1+\lambda)$ and $\frac{4}{3}\pi\rho a^3/(1+\lambda)$, or for ROCHE's problem let the latter be the mass of the particle.

Let r be the distance from the centre of one sphere to that of the other, or to the particle, as the case may be; and ω the orbital angular velocity.

In both cases we have

$$\omega^2 r^3 = \tfrac{4}{3}\pi\rho a^3.$$

The centre of inertia of the two masses is distant $r/(1+\lambda)$ and $\lambda r/(1+\lambda)$ from their respective centres, and we easily find the orbital momentum to be

$$\tfrac{4}{3}\pi\rho a^3 \omega r^2 \frac{\lambda}{(1+\lambda)^2}.$$

In both problems the rotational momentum of the first sphere is

$$\tfrac{2}{5}\left(\tfrac{4}{3}\pi\rho a^3\right) a^2 \left(\frac{\lambda}{1+\lambda}\right)^{5/3} \omega.$$

In the first problem the rotational momentum of the second sphere is

$$\tfrac{2}{5}\left(\tfrac{4}{3}\pi\rho a^3\right) a^2 \frac{1}{(1+\lambda)^{5/3}}\,\omega,$$

and in the second problem it is nil.

If, then, we write L_1 for the total angular momentum of the two spheres, and L_2 for that of the sphere and particle, we have

$$\left.\begin{aligned} L_1 &= \tfrac{4}{3}\pi\rho a^5\omega\left[\tfrac{2}{5}\frac{1+\lambda^{5/3}}{(1+\lambda)^{5/3}} + \frac{\lambda r^2}{(1+\lambda)^2 a^2}\right], \\ L_2 &= \tfrac{4}{3}\pi\rho a^5\omega\left[\tfrac{2}{5}\frac{\lambda^{5/3}}{(1+\lambda)^{5/3}} + \frac{\lambda r^2}{(1+\lambda)^2 a^2}\right]. \end{aligned}\right\} \quad \cdots \cdots \quad (1)$$

On substituting for ω its value in terms of r, these expressions become

$$L_1 = \left(\tfrac{4}{3}\pi\rho\right)^{3/2}\frac{a^5}{(1+\lambda)^2}\left[\tfrac{2}{5}\left(1+\lambda^{5/3}\right)(1+\lambda)^{1/3}\left(\frac{a}{r}\right)^{3/2} + \lambda\left(\frac{r}{a}\right)^{1/2}\right],$$

$$L_2 = \left(\tfrac{4}{3}\pi\rho\right)^{3/2}\frac{a^5}{(1+\lambda)^2}\left[\tfrac{2}{5}\lambda^{5/3}(1+\lambda)^{1/3}\left(\frac{a}{r}\right)^{3/2} + \lambda\left(\frac{r}{a}\right)^{1/2}\right].$$

To determine the minima of these functions, we differentiate with respect to r, and equate to zero.

Then, if r_1, r_2 denote the two solutions, we find

$$\left(\frac{r_1}{a}\right)^2 = \frac{6}{5\lambda}\left(1+\lambda^{5/3}\right)(1+\lambda)^{1/3},$$

$$\left(\frac{r_2}{a}\right)^2 = \tfrac{6}{5}\lambda^{2/3}(1+\lambda)^{1/3}.$$

Whence

$$\text{Minimum } L_1 = \left(\tfrac{4}{3}\pi\rho\right)^{3/2} a^5 \,.\, 4\left(\tfrac{2}{135}\right)^{1/4}\frac{\lambda^{3/4}\left(1+\lambda^{5/3}\right)^{1/4}}{(1+\lambda)^{23/12}},$$

$$\text{Minimum } L_2 = \left(\tfrac{4}{3}\pi\rho\right)^{3/2} a^5 \,.\, 4\left(\tfrac{2}{135}\right)^{1/4}\frac{\lambda^{7/6}}{(1+\lambda)^{23/12}}.$$

The ratio of the minimum of L_1 to that of L_2 is $\left(\frac{1}{\lambda^{5/3}}+1\right)^{1/4}$. Thus as λ rises from 0 to ∞ this ratio falls from infinity to unity.

All the possible cases of the first problem are comprised between $\lambda = 0$ and $\lambda = 1$. When $\lambda = 0$, $r_1 = \infty$; and when $\lambda = 1$,

$$\frac{r_1}{a} = \sqrt{\left(\tfrac{12}{5}\,.\,2^{1/3}\right)} = 1{\cdot}738.$$

Thus, in the problem of the figures of equilibrium, if one of the two masses is large compared with the other, the two must be far apart to secure secular stability. This is exactly what is to be expected from the theory of tidal friction, for limiting stability is reached when there is coalescence of the two solutions which correspond to the cases where each body always presents the same face to the other.*

The result when the two masses are equal becomes more easily intelligible when it is expressed in terms of the radius of either of them. That radius is $a/2^{1/3}$, so that when $\lambda = 1$

$$r = 1 \cdot 738 a = 2 \cdot 191 \left(\frac{a}{2^{1/3}}\right).$$

Thus, in the latter case, limiting stability is reached when the two spheres are nearly in contact with one another, for if r were equal to twice the radius of either they would be touching.

When the two bodies are far apart, the solution may be obtained by spherical harmonic analysis, and has comparatively little interest. But when the bodies are equal or nearly equal in mass, limiting stability for the figure of equilibrium would seem, from this preliminary investigation, to occur when they are quite close together. Accordingly, in finding numerical solutions hereafter, I have devoted more attention to this case than to any other.

Turning now to the solution of the analogue of ROCHE's problem, we see that when $\lambda = 0$, $r_2 = 0$. This would mean that a very small liquid satellite could be brought quite up to its planet without becoming unstable. But we shall see that, when the satellite is no longer constrainedly a sphere, instability first occurs through the variations in the shape of the satellite. This preliminary solution does not, therefore, throw much light on the matter, excepting as indicating that we must consider the cases where the satellite is as near to the planet as possible.

Next, when $\lambda = 1$, we have

$$r_2 = \sqrt{\left(\tfrac{6}{5} \cdot 2^{1/3}\right)} a = \sqrt{\tfrac{12}{5}} \cdot \left(\frac{a}{2^{1/3}}\right) = 1 \cdot 549 \left(\frac{a}{2^{1/3}}\right).$$

Thus, when the two masses are equal, their distance apart is only about $1\frac{1}{2}$ radii of either, and they will overlap. Here again it would seem as if stability would persist up to contact, but, as before, instability first sets in through variations in the shape of the satellite.

Finally, when λ is large, r_2 also becomes large. This case is the same in principle as that considered in the problem of the figures of equilibrium, for it means that if a large liquid body (formerly called the satellite) be attended by a small rigid body (formerly called the planet), secular stability will be attained when the small rigid body has been repelled by tidal friction to a great distance from the large liquid body.

* See 'Roy. Soc. Proc.,' vol. 29, 1879, p. 168, or Appendix G (b) to THOMSON and TAIT'S 'Nat. Phil.'

As this case may be adequately treated by spherical harmonic analysis, it need not detain us, and we see that the most interesting cases of ROCHE's problem are those where λ lies between 0 and 1.

§ 2. *Figures of Equilibrium of a Rotating Mass of Liquid and their Stability.*

A mass of liquid, consisting of either one or more portions, is rotating, without relative motion of its parts, about an axis through its centre of inertia with angular velocity ω. We choose as an arbitrary standard figure one which does not differ very widely from a figure of equilibrium, and we suppose that any departure from the standard figure may be defined by two parameters e and f, which may be called ellipticities. It is unnecessary to introduce more than two ellipticities, because the result for any number becomes obvious from the case of two. We also assume a definite angular velocity for the standard configuration.

Let $V(e, f)$ denote the gravitational energy lost in the concentration of the system from a condition of infinite dispersion into the configuration denoted by e, f.

Let $I(e, f), \omega(e, f)$ denote the moment of inertia and angular velocity about the axis of rotation in the same configuration.

The initial values of these quantities are those for which $e = f = 0$, and are $V(0, 0)$, $I(0, 0)$, $\omega(0, 0)$. These all refer to the arbitrary standard configuration; they are therefore constants, and I shall write them V, I, ω for brevity.

Let ellipticities e, f be imparted to the system, and let the angular velocity be so changed that the angular momentum remains constant.

Then

$$I(e, f)\,\omega(e, f) = I(0, 0)\,\omega(0, 0) = I\omega.$$

The kinetic energy of the system is half the square of the angular momentum divided by the moment of inertia; and since the angular momentum is constant it is equal to $\frac{1}{2}(I\omega)^2/I(e, f)$.

Thus the whole energy of the system, both kinetic and potential, is equal to

$$-V(e, f) + \frac{\frac{1}{2}(I\omega)^2}{I(e, f)}.$$

If $V(e, f) = V + \delta V$, $I(e, f) = I + \delta I$, the expression for the energy as far as squares of small quantities is

$$-(V + \delta V) + \frac{\frac{1}{2}(I\omega)^2}{I + \delta I} = -V + \tfrac{1}{2}I\omega^2 - \delta V - \tfrac{1}{2}\omega^2\delta I + \tfrac{1}{2}\omega^2\frac{(\delta I)^2}{I}.$$

The first two terms may be omitted as being constant and of no interest, and the energy with the sign changed, so that it is the lost energy of the system, becomes

$$\delta V + \tfrac{1}{2}\omega^2\delta I - \tfrac{1}{2}\omega^2\frac{(\delta I)^2}{I}.$$

Since ω is constant, we may write this

$$\delta\left(V+\tfrac{1}{2}\omega^2 I\right)-\tfrac{1}{2}\omega^2\frac{(\delta I)^2}{I}.$$

On developing this by TAYLOR's theorem, it becomes

$$\left(e\frac{\partial}{\partial e}+f\frac{\partial}{\partial f}+\tfrac{1}{2}e^2\frac{\partial^2}{\partial e^2}+ef\frac{\partial^2}{\partial e\partial f}+\tfrac{1}{2}f^2\frac{\partial^2}{\partial f^2}\right)\left(V+\tfrac{1}{2}\omega^2 I\right)-\tfrac{1}{2}\frac{\omega^2}{I}\left(e\frac{\partial I}{\partial e}+f\frac{\partial I}{\partial f}\right)^2.$$

The condition for a figure of equilibrium is that the first differentials of the energy with respect to the ellipticities shall vanish. If, therefore, e_0, f_0 denote the equilibrium ellipticities, the equations for finding them are

$$\left(\frac{\partial}{\partial e}+e_0\frac{\partial^2}{\partial e^2}+f_0\frac{\partial^2}{\partial e\partial f}\right)\left(V+\tfrac{1}{2}\omega^2 I\right)-\frac{\omega^2}{I}\left[e_0\left(\frac{\partial I}{\partial e}\right)^2+f_0\frac{\partial I}{\partial e}\frac{\partial I}{\partial f}\right]=0,$$

$$\left(\frac{\partial}{\partial f}+e_0\frac{\partial^2}{\partial e\partial f}+f_0\frac{\partial^2}{\partial f^2}\right)\left(V+\tfrac{1}{2}\omega^2 I\right)-\frac{\omega^2}{I}\left[e_0\frac{\partial I}{\partial e}\frac{\partial I}{\partial f}+f_0\left(\frac{\partial I}{\partial f}\right)^2\right]=0.$$

Multiplying the first of these by e and the second by f, and adding them together, we find

$$\left(e\frac{\partial}{\partial e}+f\frac{\partial}{\partial f}\right)\left(V+\tfrac{1}{2}\omega^2 I\right)=-\left[ee_0\frac{\partial^2}{\partial e^2}+(ef_0+e_0f)\frac{\partial^2}{\partial e\partial f}+ff_0\frac{\partial^2}{\partial f^2}\right]\left(V+\tfrac{1}{2}\omega^2 I\right)$$
$$+\frac{\omega^2}{I}\left[ee_0\left(\frac{\partial I}{\partial e}\right)^2+(ef_0+e_0f)\frac{\partial I}{\partial e}\frac{\partial I}{\partial f}+ff_0\left(\frac{\partial I}{\partial f}\right)^2\right].$$

On substituting this in the expression for the lost energy, it becomes

$$\left[\tfrac{1}{2}e\,(e-2e_0)\frac{\partial^2}{\partial e^2}+(ef-ef_0-e_0f)\frac{\partial^2}{\partial e\partial f}+\tfrac{1}{2}f\,(f-2f_0)\frac{\partial^2}{\partial f^2}\right]\left(V+\tfrac{1}{2}\omega^2 I\right)$$
$$-\tfrac{1}{2}\frac{\omega^2}{I}\left[e\,(e-2e_0)\left(\frac{\partial I}{\partial e}\right)^2+2\,(ef-ef_0-e_0f)\frac{\partial I}{\partial e}\frac{\partial I}{\partial f}+f\,(f-2f_0)\left(\frac{\partial I}{\partial f}\right)^2\right].$$

Now let δe, δf be the excesses of e and f above their equilibrium values e_0, f_0, so that $e=e_0+\delta e$, $f=f_0+\delta f$. Then on substitution in the expression for the lost energy it becomes

$$\left[\tfrac{1}{2}\left\{(\delta e)^2-e_0^2\right\}\frac{\partial^2}{\partial e^2}+\left\{\delta e\delta f-e_0f_0\right\}\frac{\partial^2}{\partial e\partial f}+\tfrac{1}{2}\left\{(\delta f)^2-f_0^2\right\}\frac{\partial^2}{\partial f^2}\right]\left(V+\tfrac{1}{2}\omega^2 I\right)$$
$$-\tfrac{1}{2}\frac{\omega^2}{I}\left[\left\{(\delta e)^2-e_0^2\right\}\left(\frac{\partial I}{\partial e}\right)^2+2\left\{\delta e\delta f-e_0f_0\right\}\frac{\partial I}{\partial e}\frac{\partial I}{\partial f}+\left\{(\delta f)^2-f_0^2\right\}\left(\frac{\partial I}{\partial f}\right)^2\right].$$

Since e_0, f_0 are constants, the portion of this involving e_0, f_0 explicitly is constant, and may be dropped.

Thus the variable part of the lost energy may be written

$$\tfrac{1}{2}(\delta e)^2\left[\frac{\partial^2 V}{\partial e^2}+\tfrac{1}{2}\omega^2\left(\frac{\partial^2 I}{\partial e^2}-\frac{2}{I}\left(\frac{\partial I}{\partial e}\right)^2\right)\right]+\delta e\delta f\left[\frac{\partial^2 V}{\partial e\partial f}+\tfrac{1}{2}\omega^2\left(\frac{\partial^2 I}{\partial e\partial f}-\frac{2}{I}\frac{\partial I}{\partial e}\frac{\partial I}{\partial f}\right)\right]$$
$$+\tfrac{1}{2}(\delta f)^2\left[\frac{\partial^2 V}{\partial f^2}+\tfrac{1}{2}\omega^2\left(\frac{\partial^2 I}{\partial f^2}-\frac{2}{I}\left(\frac{\partial I}{\partial f}\right)^2\right)\right].$$

This is a quadratic function of the departures of the ellipticities from their equilibrium values, and the form is obvious which the result would have if there were any number of ellipticities.

Since the condition for secular stability is that the energy shall be a minimum, the lost energy must be a maximum, and therefore this quadratic function of δe, δf, &c., must always be negative in order that the system may possess secular stability.*

If F is a quadratic function of n variables, x_1, x_2, x_3, &c., so that

$$\begin{aligned} F = a_{11}x_1^2 &+ 2a_{12}.x_1x_2+2a_{13}x_1x_3+\ldots \\ &+ a_{22}x_2^2 \quad +2a_{23}x_2x_3+\ldots \\ &\qquad\qquad + a_{33}x_3^2 \ +\ldots, \end{aligned}$$

it is known that the condition that F shall always be negative for all values of the variables is that the series of functions

$$a_{11},\quad \begin{vmatrix} a_{11}, & a_{12} \\ a_{12}, & a_{22} \end{vmatrix},\quad \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{12}, & a_{22}, & a_{23} \\ a_{13}, & a_{23}, & a_{33} \end{vmatrix},\ \ldots \qquad (2)$$

shall be alternatively negative and positive.

Since we might equally well begin with any one of the variables, it follows that a_{11}, a_{22}, $a_{33}\ldots$ must all be negative; also $a_{12}^2-a_{11}a_{22}$, $a_{13}^2-a_{11}a_{33}$, $a_{23}^2-a_{22}a_{33}\ldots$ must all be negative if F is always to be negative.

Now, suppose that F is the function of lost energy for a system with $n+1$ degrees of freedom, but that a constraint destroys one of the degrees. If the system has secular stability, the n determinants must have their appropriate signs, and when the constraint is removed, the new additional determinant must have its proper sign in order to secure secular stability. It follows that stability can never be restored by the removal of a constraint if the system was unstable when the constraint existed; but stability may be destroyed by the removal of a constraint.

* This result is also given, but with less detail, in my paper on "MACLAURIN'S Spheroid," in 'Trans. Amer. Math. Soc.,' vol. 4, No. 2, pp. 113–133 (1903).

§ 3. *On the Possibility of joining Two Masses of Liquid by a thin Neck.*

This whole investigation was undertaken principally in the hope that it might lead to an approximation to the form of the pear-shaped figure of equilibrium of a rotating mass of liquid at the stage when it should resemble an hour-glass with a thin neck. It seemed probable that such an approximation might be obtained in the following manner:—

Two masses of liquid are revolving in an orbit about one another without relative motion of their parts, so that they form a figure of equilibrium. Imagine them to be joined by a pipe without weight, through which liquid may flow from one part to the other. A flow of liquid will in general take place between the two parts, but there should be some definite partition of masses, corresponding to a given distance apart, at which flow will cease. At this stage we should have an approximation to the hour-glass figure of equilibrium.

In this section a special case of this problem is considered, in which the detached masses, to be joined by a pipe, are constrained to be spheres.

If the notation of § 1 be adopted, it is clear that the system is defined by the two parameters r and λ. In accordance with the notation of § 2 we denote the lost energy of the system by V and the moment of inertia by I. It is easily shown that

$$V = (\tfrac{4}{3}\pi\rho)^2 \mathrm{a}^5 \left[\frac{\lambda}{(1+\lambda)^2}\frac{\mathrm{a}}{r} + \tfrac{3}{5}\frac{1+\lambda^{5/3}}{(1+\lambda)^{5/3}}\right],$$

$$I = \tfrac{4}{3}\pi\rho \mathrm{a}^5 \left[\frac{\lambda}{(1+\lambda)^2}\frac{r^2}{\mathrm{a}^2} + \tfrac{2}{5}\frac{1+\lambda^{5/3}}{(1+\lambda)^{5/3}}\right].$$

For brevity write

$$F = \frac{\lambda}{(1+\lambda)^2}, \quad G = \frac{1+\lambda^{5/3}}{(1+\lambda)^{5/3}},$$

and let F', G', F'', G'' denote their first and second differentials with respect to λ.

The equations for determining the configuration of equilibrium are

$$\frac{\partial V}{\partial r} + \tfrac{1}{2}\omega^2 \frac{\partial I}{\partial r} = 0, \quad \frac{\partial V}{\partial \lambda} + \tfrac{1}{2}\omega^2 \frac{\partial I}{\partial \lambda} = 0.$$

The first of these gives at once

$$\omega^2 = \tfrac{4}{3}\pi\rho \left(\frac{\mathrm{a}}{r}\right)^3.$$

For determining the form of the second we have

$$V = (\tfrac{4}{3}\pi\rho)^2 \mathrm{a}^5 \left[F\frac{\mathrm{a}}{r} + \tfrac{3}{5}G\right], \quad I = \tfrac{4}{3}\pi\rho \mathrm{a}^5 \left[F\frac{r^2}{\mathrm{a}^2} + \tfrac{2}{5}G\right].$$

If we differentiate these with respect to λ, substitute in the second equation of equilibrium and give to ω^2 its value in terms of r^2, we find that the result is

$$\frac{a^3}{r^3}+\frac{15}{2}\frac{F'}{G'}\frac{a}{r}+3=0.$$

Now

$$F'=\frac{1-\lambda}{(1+\lambda)^3},\quad G'=\frac{5}{3}\frac{\lambda^{2/3}-1}{(1+\lambda)^{8/3}},\quad \text{and}\quad \frac{F'}{G'}=-\frac{3}{5}\frac{1+\lambda^{1/3}+\lambda^{2/3}}{(1+\lambda)^{1/3}(1+\lambda^{1/3})}.$$

Hence the equation for determining r for a given value of λ is

$$\frac{a^3}{r^3}-\frac{9}{2}\frac{1+\lambda^{1/3}+\lambda^{2/3}}{(1+\lambda)^{1/3}(1+\lambda^{1/3})}\frac{a}{r}+3=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This cubic has three real roots of which one is negative and has no physical meaning; the second gives so small a value to r that the smaller sphere is either wholly or partially inside the larger one. The third root is the one required.

In order to present the result in an easily intelligible form it may be well to express it also in terms of the radius of the larger of the two spheres, say a_1, where

$$a_1^3=\frac{a^3}{1+\lambda}.$$

The following is a table of solutions for various values of λ:—

$\lambda^{1/3}$.	r/a.	r/a_1.	$r/a_1-(1+\lambda^{1/3})$.
0·0	1·304	1·304	0·304
0·1	1·323	1·323	0·223
0·2	1·368	1·371	0·171
0·3	1·426	1·438	0·138
0·4	1·486	1·517	0·117
0·5	1·543	1·604	0·104
0·6	1·590	1·697	0·097
0·7	1·625	1·793	0·093
0·8	1·649	1·893	0·093
0·9	1·662	1·995	0·095
1·0	1·666	2·099	0·099

The solution is exhibited in fig. 1, the larger sphere being kept of constant size and the successive smaller circles representing the smaller sphere. Many of the circles pass nearly through one point, and it has not been possible to complete them without producing confusion.

The fourth column of the table gives the excess of r above the sum of the two radii of the spheres, and it shows what interval of space is unoccupied by matter. It is remarkable how nearly constant that interval is throughout a large range in the values of λ.

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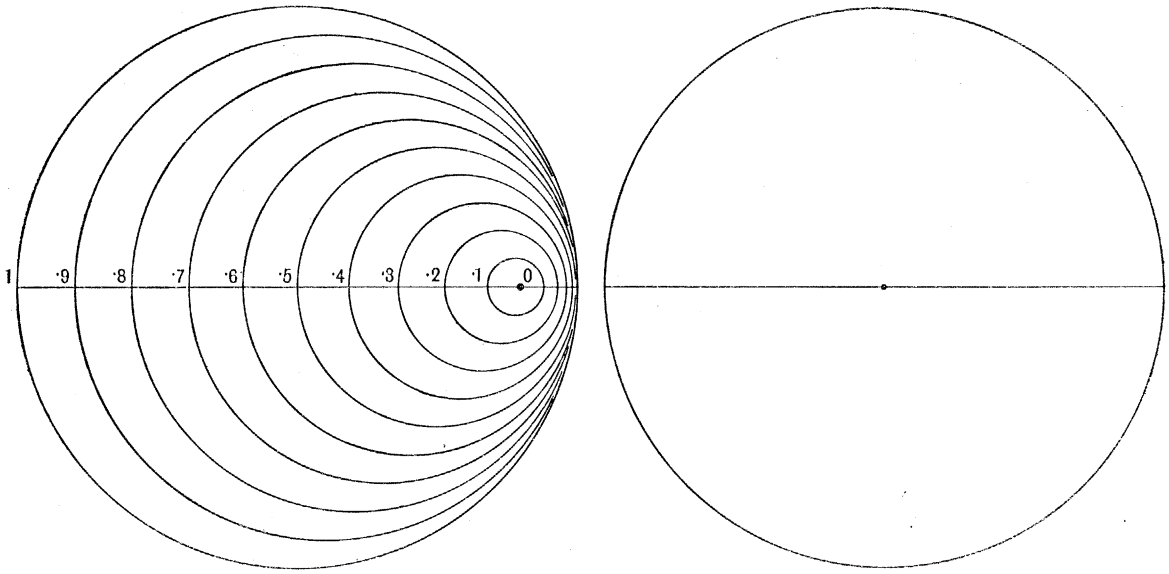

Fig. 1. Solutions for two spheres of liquid joined by a weightless pipe, for successive values of $\lambda^{1/3}$.

In the case where the two bodies are no longer spheres, the equation corresponding to the cubic (3) becomes very complicated. It is therefore desirable to discover whether in any given solution of the figure of equilibrium the two detached masses are too far apart to admit of their being joined by a weightless pipe, or whether they are too near. This may be discovered in the following way :—

Let r_0 be the solution of $f(\mathrm{a}/r_0) = 0$, where

$$f\left(\frac{\mathrm{a}}{r}\right) = \frac{\mathrm{a}^3}{r^3} - \tfrac{9}{2}\frac{1+\lambda^{1/3}+\lambda^{2/3}}{(1+\lambda)^{1/3}(1+\lambda^{1/3})}\frac{\mathrm{a}}{r} + 3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

There is only one solution of $f = 0$ between r equal to infinity and the case when the two spheres touch. Hence we can determine on which side of r_0 any given value of r lies by merely considering whether f changes from positive to negative or from negative to positive as r increases through the value r_0.

Now if $\frac{\mathrm{a}}{r_0} + \delta\left(\frac{\mathrm{a}}{r}\right)$ be any neighbouring value of $\frac{\mathrm{a}}{r}$, we have approximately

$$f\left(\frac{\mathrm{a}}{r}\right) = 3\left[\frac{\mathrm{a}^2}{r_0^{\,2}} - \tfrac{3}{2}\frac{1+\lambda^{1/3}+\lambda^{2/3}}{(1+\lambda)^{1/3}(1+\lambda^{1/3})}\right]\delta\left(\frac{\mathrm{a}}{r}\right).$$

If we express $\mathrm{a}^2/r_0^{\,2}$ in terms of r_0/a by means of the equation for r_0, this may be written

$$f\left(\frac{\mathrm{a}}{r}\right) = \frac{9}{(1+\lambda)^{1/3}}\left(1+\lambda^{1/3} - \frac{r}{\mathrm{a}_1} - \frac{\lambda^{1/3}}{1+\lambda^{1/3}}\right)\delta\left(\frac{\mathrm{a}}{r}\right),$$

where as before

$$\mathrm{a}_1^{\,3} = \frac{\mathrm{a}^3}{1+\lambda}.$$

Now the fourth column of the table shows that $1+\lambda^{1/3}-r/\mathrm{a}_1$ is negative, and the last term inside the bracket is also negative. Hence if $\delta\left(\frac{a}{r}\right)$ is negative, $f\left(\frac{a}{r}\right)$ is positive, and *vice versâ*. But if $\delta\left(\frac{a}{r}\right)$ is negative, r is greater than r_0 and the two masses are too far apart to admit of junction, and *vice versâ*.

Therefore if for a given solution for detached masses $f\left(\frac{a}{r}\right)$ is positive, the masses are too far apart to admit of junction by a weightless pipe, and if it is negative they are too near.

When in the general case we form a function $f(a/r)$, such that when the ellipticities of the two masses are annulled, it reduces to the above function, its sign will afford the criterion as to whether the masses are too far or too near to admit of junction by a thin neck of liquid. I return to this subject below in § 13.

The solution of the problem when the two masses are constrainedly spheres is so curious that it seems worth while to consider its stability. This may be done by the method of § 2.

The system depends on two parameters r and λ, and the stability will depend on three functions, which are defined as follows:—

$$\{r, r\} = \frac{\partial^2 V}{\partial r^2} + \tfrac{1}{2}\omega^2\frac{\partial^2 I}{\partial r^2} - \frac{\omega^2}{I}\left(\frac{\partial I}{\partial r}\right)^2,$$

$$\{r, \lambda\} = \frac{\partial^2 V}{\partial r \partial\lambda} + \tfrac{1}{2}\omega^2\frac{\partial^2 I}{\partial r\partial\lambda} - \frac{\omega^2}{I}\frac{\partial I}{\partial r}\frac{\partial I}{\partial\lambda},$$

$$\{\lambda, \lambda\} = \frac{\partial^2 V}{\partial\lambda^2} + \tfrac{1}{2}\omega^2\frac{\partial^2 I}{\partial\lambda^2} - \frac{\omega^2}{I}\left(\frac{\partial I}{\partial\lambda}\right)^2.$$

These functions correspond to a_{11}, a_{12}, a_{22} of (2) in § 2, and we see that for secular stability $\{r, r\}$ and $\{\lambda, \lambda\}$ must be negative, and

$$\Delta = \{r, r\}\{\lambda, \lambda\} - [\{r, \lambda\}]^2$$

must be positive.

Without giving the details of the several differentiations, I may state that if we write

$$H = (\tfrac{4}{3}\pi\rho)^2 a^3 \cdot \frac{F\frac{a^3}{r^3}}{F\frac{r^2}{a^2} + \frac{2}{5}G},$$

so that H is essentially positive, we find

$$\frac{\{r, r\}}{H} = \tfrac{6}{5}G - F\frac{r^2}{a^2}; \qquad \frac{\{r, \lambda\}}{H} = -2r\left(F'\frac{r^2}{a^2} + \tfrac{2}{5}G'\right);$$

$$\frac{\{\lambda, \lambda\}}{H} = \frac{a^2}{F}\left\{\left[\tfrac{3}{2}F''\frac{r^2}{a^2} + \tfrac{1}{5}G''\left(1 + 3\frac{r^3}{a^3}\right)\right]\left[F\frac{r^2}{a^2} + \tfrac{2}{5}G\right] - \left[F'\frac{r^2}{a^2} + \tfrac{2}{5}G'\right]^2\right\}.$$

From the equation

$$\frac{a^3}{r^3} + \tfrac{15}{2}\frac{F'}{G'}\frac{a}{r} + 3 = 0,$$

we find, after some reductions,

$$\tfrac{3}{2}F''\frac{r^2}{a^2} + \tfrac{1}{5}G''\left(1 + 3\frac{r^3}{a^3}\right) = \tfrac{3}{2}\frac{r^2}{a^2}F'\frac{d}{d\lambda}\log\frac{F'}{G'}.$$

On substituting for F', G' their values in terms of λ, I find that this expression reduces to

$$\frac{(1-\lambda^{1/3})^2}{\lambda^{1/3}(1+\lambda)^4}\frac{r^2}{a^2},$$

an essentially positive quantity.

On substitution in Δ I find

$$\Delta = \frac{H^2a^2}{F}\left(F\frac{r^2}{a^2} + \tfrac{2}{5}G\right)\left[\frac{(1-\lambda^{1/3})^2}{\lambda^{1/3}(1+\lambda)^4}\frac{r^2}{a^2}\left(\tfrac{6}{5}G - F\frac{r^2}{a^2}\right) - 3\left(F'\frac{r^2}{a^2} + \tfrac{2}{5}G'\right)^2\right]. \quad . \quad . \quad (5)$$

The factors outside [ ] are essentially positive and do not affect the sign of Δ, and it is clear that Δ can only be positive if $\tfrac{6}{5}G - F\frac{r^2}{a^2}$ is positive. But Δ must be positive for secular stability; hence stability can only be secured by $\tfrac{6}{5}G - F\frac{r^2}{a^2}$ being positive, and it is not necessarily so secured. But if this function is positive, so also is $\{r, r\}$, and if this last is positive the system is unstable. Hence stability is always impossible. As a fact, in all the solutions given above $\{r, r\}$ is positive, and we should have to move the spheres much further apart to make it negative, and therefore on this ground alone the system is always unstable. But Δ is sometimes positive and sometimes negative and vanishes for a certain value of λ. As the vanishing of Δ puzzled me a good deal, I propose to examine the matter further.

Before doing so, however, I will show that the instability of the system may be concluded from other considerations.

It was proved in § 1 that two spheres, unconnected by a pipe, are in limiting stability when their distance apart is given by

$$\frac{r^2}{a^2} = \frac{6}{5\lambda}(1+\lambda^{5/3})(1+\lambda)^{1/3} = \tfrac{6}{5}\frac{G}{F}.$$

This is the condition that $\{r, r\}$ should vanish.

When λ is zero the two spheres in limiting stability are infinitely far apart, and when λ is unity they are as near as possible, and $r = 1{\cdot}738a$.

Now the table of solutions in the case where the two are connected by a pipe shows that they are furthest apart when λ is unity, and that then $r = 1{\cdot}666a$.

The removal of the constraint of one degree of freedom may destroy stability, but cannot create it. Hence, when two spheres revolve about one another, the opening of a channel of communication between them may destroy stability, but cannot create it. When two equal spheres revolve about one another at such a distance that they could be connected by a pipe and yet remain in equilibrium, their distance is 1·666; but they are then unstable, because 1·666 is less than 1·738. The opening of a pipe between them, being the removal of a constraint, cannot make the motion stable. *À fortiori* the like is true when the two spheres are unequal in mass.

Hence the system of equilibrium of two spheres joined by a pipe is unstable in all cases.

I will now consider the meaning of the vanishing of Δ.

Having evaluated the angular momentum of the system corresponding to the several solutions tabulated above, I found it had a minimum when $\lambda^{1/3} = 0·254$. Such a solution is a critical one and is the starting point of two solutions of which one must have one fewer degrees of instability than the other. The vanishing of Δ must have the same meaning, but it remains to be proved that minimum angular momentum is secured by the vanishing of Δ.

The angular momentum is $I\omega$, and is therefore proportional to μ, where

$$\mu = F\left(\frac{r}{a}\right)^{1/2} + \tfrac{2}{5}G\left(\frac{a}{r}\right)^{3/2}.$$

On equating $\dfrac{d\mu}{d\lambda}$ to zero so as to find its minimum, we have

$$2r\left(F'\frac{r^2}{a^2}+\tfrac{2}{5}G'\right)+\left(F\frac{r^2}{a^2}-\tfrac{6}{5}G\right)\frac{dr}{d\lambda}=0.$$

Now since

$$\frac{r^3}{a^3}+\tfrac{1}{3}+\tfrac{5}{2}\frac{F'}{G'}\frac{r}{a}=0,$$

we have

$$\frac{dr}{d\lambda}=\frac{-\tfrac{1}{3}r\,(1-\lambda^{1/3})^2\,\lambda^{-1/3}\,(1+\lambda)^{-4}}{\tfrac{3}{5}G'\dfrac{r}{a}+F'}.$$

On substituting this value in the equation $d\mu/d\lambda = 0$, I find that the result may be written

$$\frac{(1-\lambda^{1/3})^2}{\lambda^{1/3}(1+\lambda)^4}\frac{r^2}{a^2}\left(\tfrac{6}{5}G-F\frac{r^2}{a^2}\right)+6\frac{r^2}{a^2}\left(F'\frac{r^2}{a^2}+\tfrac{2}{5}G'\right)\left(\tfrac{3}{5}G'\frac{r}{a}+F'\right)=0.$$

The first term of this is the same as the first term inside the bracket in the expression for Δ in (5). On comparing the two second terms together we see that $\Delta = 0$ is the condition for minimum angular momentum, if

$$6\frac{r^2}{a^2}\left(F'\frac{r^2}{a^2}+\tfrac{2}{5}G'\right)\left(\tfrac{3}{5}G'\frac{r}{a}+F'\right)=-3\left(F'\frac{r^2}{a^2}+\tfrac{2}{5}G'\right)^2;$$

that is to say, if

$$2\frac{r^2}{\mathrm{a}^2}\left(\tfrac{3}{5}G'\frac{r}{\mathrm{a}}+F'\right)+F'\frac{r^2}{\mathrm{a}^2}+\tfrac{2}{5}G'=0,$$

or if

$$\frac{\mathrm{a}^3}{r^3}+\tfrac{15}{2}\frac{F'}{G'}\frac{\mathrm{a}}{r}+3=0.$$

But this last is true, being the equation (3) determining the figure of equilibrium; hence $\Delta = 0$ gives minimum angular momentum.

Since two liquid spheres cannot be joined stably by a pipe, it seems very improbable that two tidal ellipsoids could be so joined as to become stable. Indeed, if the distortion of the surfaces of the two masses into ellipsoidal forms may be regarded as due to the removal of constraints whereby they were previously maintained in a spherical form, stability is impossible.

The question as to whether or not there is an unstable figure with a thin neck will be considered later, for the present we are only concerned with the conclusion that there is no stable figure of this kind.

Mr. Jeans has treated an analogous problem in his paper on the equilibrium of rotating liquid cylinders,* and has concluded that the cylinder will divide stably into two portions. The analogy is so close between his problem and the three-dimensional case, that it might have been expected that the analogy would subsist throughout; nevertheless, if we are both correct there must be a divergence between them at some point.

§ 4. *Notation.*

As the solution given below is effected by means of ellipsoidal harmonic analysis, it is well to state the notation employed. It is that used in four previous papers to which references are given in the Preface.

In "Harmonics" the squares of the semi-axes of the ellipsoid were

$$a^2 = k^2\left(\nu^2-\frac{1+\beta}{1-\beta}\right), \quad b^2 = k^2(\nu^2-1), \quad c^2 = k^2\nu^2.$$

The rectangular co-ordinates were connected with ellipsoidal co-ordinates ν, μ, ϕ by

$$\frac{x^2}{k^2} = -\frac{1-\beta}{1+\beta}\left(\nu^2-\frac{1+\beta}{1-\beta}\right)\left(\mu^2-\frac{1+\beta}{1-\beta}\right)\cos^2\phi,$$

$$\frac{y^2}{k^2} = -(\nu^2-1)(\mu^2-1)\sin^2\phi,$$

$$\frac{z^2}{k^2} = \nu^2\mu^2\frac{1-\beta\cos 2\phi}{1+\beta}.$$

* 'Phil. Trans. Roy. Soc.,' Series A, vol. 200, pp. 67–104.

The three roots of the cubic

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$$

were

$$u_1=k^2\nu^2, \quad u_2=k^2\mu^2, \quad u_3=k^2\frac{1-\beta\cos 2\phi}{1-\beta}.$$

Lastly ν ranges from ∞ to 0, μ between $\pm$ 1, ϕ from 0 to 2π.

In the two later papers, I put

$$\kappa^2=\frac{1-\beta}{1+\beta}, \quad \kappa'^2=1-\kappa^2, \quad \nu=\frac{1}{\kappa\sin\gamma}, \quad \mu=\sin\theta;$$

and for convenience I introduced an auxiliary constant β (easily distinguishable from the β of the previous notation) defined by $\sin\beta=\kappa\sin\gamma$.

The squares of the semi-axes of the ellipsoid were then

$$a^2=\frac{k^2\cos^2\gamma}{\sin^2\beta}, \quad b^2=\frac{k^2\cos^2\beta}{\sin^2\beta}, \quad c^2=\frac{k^2}{\sin^2\beta}.$$

The rectangular co-ordinates became

$$\frac{x^2}{k^2}=\frac{\cos^2\gamma}{\sin^2\beta}(1-\kappa^2\sin^2\theta)\cos^2\phi, \quad \frac{y^2}{k^2}=\frac{\cos^2\beta}{\sin^2\beta}\cos^2\theta\sin^2\phi, \quad \frac{z^2}{k^2}=\frac{1}{\sin^2\beta}\sin^2\theta(1-\kappa'^2\cos^2\phi).$$

The roots of the cubic were

$$u_1=\frac{k^2}{\sin^2\beta}, \quad u_2=k^2\sin^2\theta, \quad u_3=\frac{k^2}{\kappa^2}(1-\kappa'^2\cos^2\phi).$$

The notation employed for the harmonic functions is that defined in "Harmonics."

§ 5. *The Determination of Gravity on* Roche's *Ellipsoid.*

In Roche's problem a mass of liquid, which assumes approximately the form of an ellipsoid, revolves in a circular orbit about a distant centre of force without any relative motion. In the present section it is proposed to evaluate gravity on the surface of this ellipsoid. I intend to solve the problems of the present paper by means of the principles of energy, and for that purpose it is necessary to determine the law of gravity.

Suppose that the ellipsoid of reference, defined by ν_0, is deformed by a normal displacement defined by the function $pf(\mu, \phi)$, where p is the perpendicular from the centre on to the tangent plane at μ, ϕ. This deformation must be expressible by a series of ellipsoidal harmonic functions, and therefore we may assume

$$f(\mu,\phi)=\Sigma e_i^s \mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi).$$

The typical term written down must be deemed to include sine-functions as well as cosine-functions, and all those types which I have denoted by **P**, **C**, **S** in "Harmonics."

On multiplying each side of our equation by any harmonic function, and integrating over the surface of the ellipsoid, an element of which surface is denoted by $d\sigma$, we find in the usual way

$$e_i^s = \frac{\int f(\mu, \phi)\, \mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi)\, p\, d\sigma}{\int [\mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi)]^2\, p\, d\sigma}.$$

Suppose that $f(\mu, \phi)$ is zero everywhere except over a small area $\delta\alpha$ situated at the point μ', ϕ', and that it is there equal to a constant c; also let p' be the value of p at this area $\delta\sigma$.

Then the mass of the inequality is

$$\int \rho f(\mu, \phi)\, p d\sigma = cp'\rho\delta\alpha,$$

where ρ is the density of the solid ellipsoid which is deformed.

Next let us suppose that the mass of the inequality is unity, so that

$$cp'\rho\delta\alpha = 1.$$

Then we have

$$\int f(\mu, \phi)\, \mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi)\, p\, d\sigma = c\mathfrak{P}_i^s(\mu')\, \mathfrak{C}_i^s(\phi')\, p'\delta\alpha = \frac{1}{\rho}\mathfrak{P}_i^s(\mu')\, \mathfrak{C}_i^s(\phi').$$

Hence

$$e_i^s = \frac{\mathfrak{P}_i^s(\mu')\, \mathfrak{C}_i^s(\phi')}{\rho \int [\mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi)]^2\, p\, d\sigma}.$$

I now write M for the mass of the ellipsoid, and shall subsequently make it equal to $\frac{4}{3}\pi\rho a^3 \frac{\lambda}{1+\lambda}$, while the mass of the distant particle will be $\frac{M}{\lambda}$ or $\frac{4}{3}\pi\rho a^3 \frac{1}{1+\lambda}$.

Since $\int p\rho\, d\sigma = 3M$, and $\mathfrak{P}_0(\mu)\, \mathfrak{C}_0(\phi) = 1$, we have $e_0 = \frac{1}{3M}$.

Thus an inequality representing a particle of unit mass at μ', ϕ' on the surface of the ellipsoid is expressed in ellipsoidal harmonics by

$$p\left[\frac{1}{3M} + \Sigma \frac{\mathfrak{P}_i^s(\mu')\, \mathfrak{C}_i^s(\phi')\, \mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi)}{\rho \int [\mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi)]^2\, p\, d\sigma}\right].$$

By the formula (51) of "Harmonics," the external potential at the point ν, μ, ϕ of the inequality is

$$\frac{1}{k}\mathfrak{P}_0(\nu_0)\, \mathfrak{Q}_0(\nu_0) + \frac{3M}{k\rho}\Sigma \frac{\mathfrak{P}_i^s(\mu')\, \mathfrak{C}_i^s(\phi')\, \mathfrak{P}_i^s(\nu_0)\, \mathfrak{Q}_i^s(\nu)\, \mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi)}{\int [\mathfrak{P}_i^s(\mu)\, \mathfrak{C}_i^s(\phi)]^2\, p\, d\sigma}.$$

But if R is the distance between the point ν_0, μ', ϕ' on the ellipsoid and the external point ν, μ, ϕ, this potential is $1/R$.

If we imagine a particle of mass M/λ situated at ν, μ, ϕ, the above expression multiplied by M/λ is the potential of the particle at the point ν_0, μ', ϕ' on the ellipsoid.

We have no need for the general expression for the potential of a particle situated anywhere in space at the surface of the ellipsoid, because it is only necessary to consider the case where the particle lies on the prolongation of the longest axis of the ellipsoid. In this case

$$\mu = 1, \quad \phi = \tfrac{1}{2}\pi, \quad \nu = \frac{r}{k},$$

where r is the distance of the particle from the centre of the ellipsoid.

But it is now no longer necessary to retain the accents to μ', ϕ', since they are only the co-ordinates of a point on the ellipsoid.

Thus the potential of M/λ, lying on the longest axis of the ellipsoid at a distance r from the centre, at the point ν_0, μ, ϕ on the ellipsoid, is

$$\frac{M}{k\lambda}\mathfrak{P}_0(\nu_0)\,\mathfrak{Q}_0\left(\frac{r}{k}\right) + \frac{3M^2}{k\lambda\rho}\Sigma\frac{\mathfrak{Q}_i^s\left(\frac{r}{k}\right)\mathfrak{P}_i^s(1)\,\mathfrak{C}_i^s(\tfrac{1}{2}\pi)\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)}{\int[\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)]^2\,p\,d\sigma}.$$

For the types of functions denoted in "Harmonics" EES, OOS, OES, EOS, $\mathfrak{P}_i^s(1) = 0$, and for EOC, OOC, $\mathfrak{C}_i^s(\tfrac{1}{2}\pi) = 0$. The only types for which $\mathfrak{P}_i^s(1)\,\mathfrak{C}_i^s(\tfrac{1}{2}\pi)$ does not vanish are EEC, OEC; that is to say, cosine-functions of even rank. Accordingly the functions left are $\mathfrak{P}_i^s\mathfrak{C}_i^s$ for i and s even, and $\mathfrak{P}_i^s\mathbf{C}_i^s$ for i odd and s even; we may however continue to allow $\mathfrak{C}_i^s$ to stand for both types.

For brevity write

$$\mathfrak{T}_i^s = \frac{(2i+1)\,\rho}{3M}\int[\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)]^2\,p\,d\sigma.$$

Thence, since $\mathfrak{P}_0(\nu_0) = 1$, the potential may be written

$$\frac{M}{k\lambda}\left\{\mathfrak{Q}_0\left(\frac{r}{k}\right) + \sum_1^\infty \frac{2i+1}{\mathfrak{T}_i^s}\,\mathfrak{Q}_i^s\left(\frac{r}{k}\right)\mathfrak{P}_i^s(1)\,\mathfrak{C}_i^s(\tfrac{1}{2}\pi)\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)\right\} \text{ for all even values of } s.$$

It must be observed that $\mathfrak{P}_i^s$ and $\mathfrak{C}_i^s$ occur as squares in $\mathfrak{T}_i^s$; they also occur twice in the numerator in the forms $\mathfrak{P}_i^s(1)\,\mathfrak{P}_i^s(\mu)$ and $\mathfrak{C}_i^s(\tfrac{1}{2}\pi)\,\mathfrak{C}_i^s(\phi)$.

Again $\mathfrak{Q}_i^s\left(\frac{r}{k}\right)$ is of dimensions -1 in $\mathfrak{P}_i^s$, and therefore $\mathfrak{Q}_i^s\left(\frac{r}{k}\right)\mathfrak{P}_i^s(\nu_0)$ is of zero dimensions. From these considerations it follows that $\mathfrak{P}_i^s(\mu)$ and $\mathfrak{C}_i^s(\phi)$ may be multiplied by any factors without changing the result, and further that $\mathfrak{P}_i^s(\nu)$ may differ in its mode of definition from $\mathfrak{P}_i^s(\mu)$ without producing any change.

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The higher harmonics will be considered later, and for the present it is only necessary to consider the terms defined by $i = 1$, $s = 0$ and $i = 2$, $s = 0$ and 2.

The following are the definitions of the several functions, in accordance with "Integrals":—

$$\mathfrak{P}_0(\nu) = 1, \qquad \mathfrak{P}_0(\mu) = 1, \qquad \mathfrak{C}_0(\phi) = 1,$$
$$\mathfrak{P}_1(\nu) = \nu, \qquad \mathfrak{P}_1(\mu) = \mu, \qquad \mathbf{C}_1(\phi) = \sqrt{(1-\kappa'^2\cos^2\phi)},$$
$$\mathfrak{P}_2^s(\nu) = \nu^2 - \frac{q_s^2}{\kappa^2}, \quad \mathfrak{P}_2^s(\mu) = \kappa^2\mu^2 - q_s^2, \quad \mathfrak{C}_2^s(\phi) = q_s'^2 - \kappa'^2\cos^2\phi \quad (s = 0, 2),$$

where $q_s^2 = \frac{1}{3}[1+\kappa^2 \mp D]$ and $D^2 = 1-\kappa^2\kappa'^2$, with upper sign for $s = 0$, and lower for $s = 2$.

Hence

$$\mathfrak{P}_1(1)\,\mathfrak{C}_1(\tfrac{1}{2}\pi) = 1, \quad \mathfrak{P}_2^s(1)\,\mathfrak{C}_2^s(\tfrac{1}{2}\pi) = (\kappa^2 - q_s^2)\,q_s'^2 \quad (s = 0, 2).$$

Then from "Integrals," equations (5) and (6),

$$\mathfrak{T}_1 = 1, \quad \mathfrak{T}_2^s = \frac{2^3}{3^4}[D^4 \pm (1+\tfrac{1}{2}\kappa^2\kappa'^2)(1-2\kappa'^2)D] \quad (s = 0, 2).$$

Thus as far as the second order of harmonics the potential of M/λ at ν_0, μ, ϕ is

$$\frac{M}{k\lambda}\left\{\mathfrak{Q}_0\left(\frac{r}{k}\right) + 3\mathfrak{Q}_1\left(\frac{r}{k}\right)\mathfrak{P}_1(\nu_0)\,\mathfrak{P}_1(\mu)\,\mathbf{C}_1(\phi) + \frac{5}{\mathfrak{T}_2}(\kappa^2 - q_0^2)\,q_0'^2\mathfrak{Q}_2\left(\frac{r}{k}\right)\mathfrak{P}_2(\nu_0)\,\mathfrak{P}_2(\mu)\,\mathfrak{C}_2(\phi)\right.$$
$$\left. + \frac{5}{\mathfrak{T}_2^2}(\kappa^2 - q_2^2)\,q_2'^2\mathfrak{Q}_2^2\left(\frac{r}{k}\right)\mathfrak{P}_2^2(\nu_0)\,\mathfrak{P}_2^2(\mu)\,\mathfrak{C}_2^2(\phi)\right\}.$$

We must now express the several solid harmonics involved in this expression in terms of x, y, z co-ordinates of a point on the surface of the ellipsoid.

We have

$$\mathfrak{P}_1(\nu_0)\,\mathfrak{P}_1(\mu)\,\mathbf{C}_1(\phi) = \nu\mu\sqrt{(1-\kappa'^2\cos^2\phi)} = \frac{z}{k}.$$

By the definition of ellipsoidal co-ordinates the three values of ω^2 which satisfy the equation

$$\frac{x^2}{\omega^2 - 1/\kappa^2} + \frac{y^2}{\omega^2 - 1} + \frac{z^2}{\omega^2} - k^2 = 0 \quad \text{are} \quad \nu^2,\ \mu^2,\ \frac{1}{\kappa^2}(1-\kappa'^2\cos^2\phi).$$

Hence we have the following identity

$$\frac{x^2}{\omega^2 - 1/\kappa^2} + \frac{y^2}{\omega^2 - 1} + \frac{z^2}{\omega^2} - k^2 = k^2\frac{(\nu_0^2 - \omega^2)(\mu^2 - \omega^2)(1 - \kappa'^2\cos^2\phi - \omega^2\kappa^2)}{(\omega^2 - 1/\kappa^2)(\omega^2 - 1)\,\omega^2\kappa^2}.$$

Putting $\omega^2 = \dfrac{q_s^2}{\kappa^2}$ $(s = 0, 2)$ we find

$$\mathfrak{P}_2^s(\nu_0)\,\mathfrak{P}_2^s(\mu)\,\mathfrak{C}_2^s(\phi) = q_s^2 q_s'^2(\kappa^2 - q_s^2)\left[-\frac{1}{q_s'^2}\cdot\frac{x^2}{k^2} - \frac{1}{\kappa^2 - q_s^2}\cdot\frac{y^2}{k^2} + \frac{1}{q_s^2}\cdot\frac{z^2}{k^2} - \frac{1}{\kappa^2}\right] (s = 0, 2).$$

Hence the potential of M/λ is

$$\frac{M}{\lambda k}\left\{\mathfrak{Q}_0\left(\frac{r}{k}\right)+3\mathfrak{Q}_1\left(\frac{r}{k}\right)\cdot\frac{z}{k}\right.$$

$$+\frac{x^2}{k^2}\left[-\frac{5\mathfrak{Q}_2\left(\frac{r}{k}\right)}{\mathfrak{T}_2}q_0^2q_0'^2(\kappa^2-q_0^2)^2-\frac{5\mathfrak{Q}_2^2\left(\frac{r}{k}\right)}{\mathfrak{T}_2^2}q_2^2q_2'^2(\kappa^2-q_2^2)^2\right]$$

$$+\frac{y^2}{k^2}\left[-\frac{5\mathfrak{Q}_2\left(\frac{r}{k}\right)}{\mathfrak{T}_2}q_0^2q_0'^4(\kappa^2-q_0^2)-\frac{5\mathfrak{Q}_2^2\left(\frac{r}{k}\right)}{\mathfrak{T}_2^2}q_2^2q_2'^4(\kappa^2-q_2^2)\right]$$

$$+\frac{z^2}{k^2}\left[\frac{5\mathfrak{Q}_2\left(\frac{r}{k}\right)}{\mathfrak{T}_2}q_0'^4(\kappa^2-q_0^2)^2+\frac{5\mathfrak{Q}_2^2\left(\frac{r}{k}\right)}{\mathfrak{T}_2^2}q_2'^4(\kappa^2-q_2^2)^2\right]$$

$$\left.-\frac{1}{\kappa^2}\left[\frac{5\mathfrak{Q}_2\left(\frac{r}{k}\right)}{\mathfrak{T}_2}q_0^2q_0'^4(\kappa^2-q_0^2)^2+\frac{5\mathfrak{Q}_2^2\left(\frac{r}{k}\right)}{\mathfrak{T}_2^2}q_2^2q_2'^4(\kappa^2-q_2^2)^2\right]\right\}.$$

For the object immediately in view we only need the terms involving x, y, z, and may therefore drop the first and last terms.

The expressions for q_s^2 and for $\mathfrak{T}_2^s$ in terms of κ^2 have been given above; by means of these I find that

$$\frac{q_0'^2(\kappa^2-q_0^2)}{\mathfrak{T}_2}=\frac{9q_2^2}{4D\kappa^2},\quad \frac{q_2'^2(\kappa^2-q_2^2)}{\mathfrak{T}_2^2}=-\frac{9q_0^2}{4D\kappa^2}$$

(note the interchange in the suffixes of the q's).

A common factor $\frac{45}{4}\frac{q_0^2q_2^2}{D\kappa^2}$ may be taken from all the coefficients of x^2, y^2, z^2, and since $q_0^2q_2^2=\frac{1}{3}\kappa^2$ this common factor is equal to $15/4D$. Hence the terms in x^2, y^2, z^2 inside { } become

$$+\frac{15}{4D}\frac{x^2}{k^2}\left[-\mathfrak{Q}_2\left(\frac{r}{k}\right)q_0'^2(\kappa^2-q_0^2)+\mathfrak{Q}_2^2\left(\frac{r}{k}\right)q_2'^2(\kappa^2-q_2^2)\right]$$

$$+\frac{15}{4D}\frac{y^2}{k^2}\left[-\mathfrak{Q}_2\left(\frac{r}{k}\right)q_0'^2+\mathfrak{Q}_2^2\left(\frac{r}{k}\right)q_2'^2\right]$$

$$+\frac{15}{4D}\frac{z^2}{k^2}\left[\mathfrak{Q}_2\left(\frac{r}{k}\right)\frac{q_0'^2}{q_0^2}(\kappa^2-q_0^2)-\mathfrak{Q}_2^2\left(\frac{r}{k}\right)\frac{q_2'^2}{q_2^2}(\kappa^2-q_2^2)\right].$$

On substituting for the several coefficients their values in terms of κ, I find that the potential of M/λ may be written in the following form:—

$$\frac{M}{k\lambda}\left\{3\mathfrak{Q}_1\left(\frac{r}{k}\right)\cdot\frac{z}{k}+\tfrac{5}{4}\frac{x^2}{k^2}\left[-(2\kappa^2-1)\frac{\mathfrak{Q}_2-\mathfrak{Q}_2^{\,2}}{D}-(\mathfrak{Q}_2+\mathfrak{Q}_2^{\,2})\right]\right.$$
$$+\tfrac{5}{4}\frac{y^2}{k^2}\left[-(2-\kappa^2)\frac{\mathfrak{Q}_2-\mathfrak{Q}_2^{\,2}}{D}-(\mathfrak{Q}_2+\mathfrak{Q}_2^{\,2})\right]$$
$$\left.+\tfrac{5}{4}\frac{z^2}{k^2}\left[(1+\kappa^2)\frac{\mathfrak{Q}_2-\mathfrak{Q}_2^{\,2}}{D}+2(\mathfrak{Q}_2+\mathfrak{Q}_2^{\,2})\right]\right\} \quad . \quad . \quad . \quad . \quad (6).$$

It may be observed that this satisfies LAPLACE's equation, as it should do.

It remains to obtain approximate expressions for the $\mathfrak{Q}$'s.

The expression for these functions is given by

$$\mathfrak{Q}_i^s(\nu)=\mathfrak{P}_i^s(\nu)\int_\nu^\infty\frac{d\nu}{[\mathfrak{P}_i^s(\nu)]^2(\nu^2-1)^{1/2}(\nu^2-1/\kappa^2)^{1/2}}.$$

We require these when ν, which will be put equal to r/k, is large; thus we must develop in powers of $1/\nu$.

Now

$$(\nu^2-1)^{-1/2}(\nu^2-1/\kappa^2)^{-1/2}=\frac{1}{\nu^2}\left[1+\frac{1+\kappa^2}{2\kappa^2\nu^2}+\frac{3+2\kappa^2+3\kappa^4}{2^3\kappa^4\nu^4}+\frac{5+3\kappa^2+3\kappa^4+5\kappa^6}{2^4\kappa^6\nu^6}+\ldots\right].$$

Since $\mathfrak{P}_0(\nu)=1$, we have by integration

$$\mathfrak{Q}_0\left(\frac{r}{k}\right)=\frac{k}{r}\left[1+\frac{1+\kappa^2}{6\kappa^2}\cdot\frac{k^2}{r^2}+\frac{3+2\kappa^2+3\kappa^4}{40\kappa^4}\cdot\frac{k^4}{r^4}+\frac{5+3\kappa^2+3\kappa^4+5\kappa^6}{112\kappa^6}\cdot\frac{k^6}{r^6}+\ldots\right]. \quad (7).$$

There is no immediate need for this term, since it has been omitted above, but it will occur again hereafter.

Since $\mathfrak{P}_1(\nu)=\nu$, we have

$$3\mathfrak{Q}_1\left(\frac{r}{k}\right)=\frac{k^2}{r^2}\left[1+\frac{3(1+\kappa^2)}{10\kappa^2}\frac{k^2}{r^2}+\frac{3(3+2\kappa^2+3\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}+\ldots\right]. \quad . \quad . \quad . \quad (8).$$

Lastly, since $\mathfrak{P}_2^s(\nu)=\nu^2-q_s^2/\kappa^2$, we have

$$[\mathfrak{P}_2^s(\nu)]^{-2}=\frac{1}{\nu^4}\left[1+\frac{2q_s^{\,2}}{\kappa^2\nu^2}+\frac{3q_s^{\,4}}{\kappa^4\nu^4}+\ldots\right] \quad (s=0,\ 2),$$

so that

$$\frac{1}{[\mathfrak{P}_2^s(\nu)]^2(\nu^2-1)^{1/2}(\nu^2-1/\kappa^2)^{1/2}}$$
$$=\frac{1}{\nu^6}\left[1+\left(\frac{1+\kappa^2}{2\kappa^2}+\frac{2q_s^2}{\kappa^2}\right)\frac{1}{\nu^2}+\left(\frac{3+2\kappa^2+3\kappa^4}{8\kappa^4}+\frac{(1+\kappa^2)q_s^{\,2}}{\kappa^4}+\frac{3q_s^{\,4}}{\kappa^4}\right)\frac{1}{\nu^4}+\ldots\right](s=0,\ 2).$$

If we integrate this, multiply it by $\mathfrak{P}_2^s(\nu)$ and write r/k for ν, we find,

$$5\mathfrak{Q}_2^s\left(\frac{r}{k}\right)=\frac{k^3}{r^3}\left[1+\left\{\frac{5(1+\kappa^2)}{14\kappa^2}+\frac{3q_s^{\,2}}{7\kappa^2}\right\}\frac{k^2}{r^2}+5\left\{\frac{3+2\kappa^2+3\kappa^4}{72\kappa^4}+\frac{5q_s^{\,2}}{126\kappa^4}+\frac{q_s^{\,4}}{21\kappa^4}\right\}\frac{k^4}{r^4}\ldots\right](s=0,2) \quad (9).$$

On substituting for q_s its value, I find

$$5\mathfrak{Q}_2^s\left(\frac{r}{k}\right) = \frac{k^3}{r^3}\left[1 + \frac{7(1+\kappa^2)\mp 2D}{14\kappa^2}\frac{k^2}{r^2} + \frac{5\{11+10\kappa^2+11\kappa^4\mp 4(1+\kappa^2)D\}}{168\kappa^4}\frac{k^4}{r^4}\cdots\right].$$

Whence

$$5[\mathfrak{Q}_2+\mathfrak{Q}_2^2] = \frac{2k^3}{r^3}\left[1 + \frac{1+\kappa^2}{2\kappa^2}\frac{k^2}{r^2} + \frac{5(11+10\kappa^2+11\kappa^4)}{168\kappa^4}\frac{k^4}{r^4}\cdots\right],$$

$$\frac{5}{D}[\mathfrak{Q}_2-\mathfrak{Q}_2^2] = -\frac{2k^3}{r^3}\left[\frac{1}{7\kappa^2}\frac{k^2}{r^2} + \frac{20(1+\kappa^2)}{168\kappa^4}\frac{k^4}{r^4}\cdots\right].$$

Substituting these values in (6) we have for the potential of M/λ at the surface of the ellipsoid, as far as concerning terms involving x, y, z,

$$\begin{aligned}\frac{M}{\lambda k}\Bigg\{3\mathfrak{Q}_1\left(\frac{r}{k}\right)\cdot\frac{z}{k} - \frac{x^2}{2k^2}\cdot\frac{k^3}{r^3}\left[1 + \frac{3(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2} + \frac{5(5+2\kappa^2+\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}\cdots\right]\\ - \frac{y^2}{2k^2}\cdot\frac{k^3}{r^3}\left[1 + \frac{3(1+3\kappa^2)}{14\kappa^2}\frac{k^2}{r^2} + \frac{5(5\kappa^4+2\kappa^2+1)}{56\kappa^4}\frac{k^4}{r^4}\cdots\right]\\ + \frac{z^2}{k^2}\cdot\frac{k^3}{r^3}\left[1 + \frac{3(1+\kappa^2)}{7\kappa^2}\frac{k^2}{r^2} + \frac{5(3+2\kappa^2+3\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}\cdots\right]\Bigg\} \quad . \quad . \quad (10).\end{aligned}$$

If the system be rendered statical by the imposition of a rotation potential, we must add to the above such a potential, and that of the ellipsoid itself.

The expression for the internal potential of an ellipsoid ν_0 is given in (65) of "Harmonics"; it is

$$\frac{3M}{2k}\left\{\frac{\mathfrak{Q}_0(\nu_0)}{\mathfrak{P}_0(\nu_0)} - \frac{x^2}{k^2}\frac{\mathsf{Q}_1^1(\nu_0)}{\mathsf{P}_1^1(\nu_0)} - \frac{y^2}{k^2}\frac{\mathfrak{Q}_1^1(\nu_0)}{\mathfrak{P}_1^1(\nu_0)} - \frac{z^2}{k^2}\frac{\mathfrak{Q}_1(\nu_0)}{\mathfrak{P}_1(\nu_0)}\right\}.$$

I will now introduce an abridged notation which was used in some of my previous papers, as follows:-

$$\mathsf{P}_1^1(\nu_0)\,\mathsf{Q}_1^1(\nu_0) = \mathsf{A}_1^1, \quad \mathfrak{P}_1^1(\nu_0)\,\mathfrak{Q}_1^1(\nu_0) = \mathfrak{A}_1^1, \quad \mathfrak{P}_1(\nu_0)\,\mathfrak{Q}_1(\nu_0) = \mathfrak{A}_1.$$

Then, since

$$\mathsf{P}_1^1(\nu_0) = \sqrt{\left(\nu_0^2-\frac{1}{\kappa^2}\right)}, \quad \mathfrak{P}_1^1(\nu_0) = \sqrt{(\nu_0^2-1)}, \quad \mathfrak{P}_1(\nu_0) = \nu_0,$$

we may, on omitting the term independent of x, y, z, write this potential in the form

$$-\frac{3M}{2k}\left[\frac{x^2}{k^2(\nu_0^2-1/\kappa^2)}\mathsf{A}_1^1 + \frac{y^2}{k^2(\nu_0^2-1)}\mathfrak{A}_1^1 + \frac{z^2}{k^2\nu_0^2}\mathfrak{A}_1\right] \quad . \quad . \quad . \quad . \quad (11)$$

The rotation with angular velocity ω takes place about an axis parallel to x through the centre of inertia of the system, which consists of two masses M and M/λ distant r from one another. Hence the rotation potential is

$$\tfrac{1}{2}\omega^2\left[y^2+\left(z-\frac{r}{1+\lambda}\right)^2\right] = \frac{3M}{2k}\left\{\frac{\omega^2k^3}{3M}\left(\frac{y^2+z^2}{k^2}\right) - \frac{2}{3}\frac{\omega^2k^2r}{(1+\lambda)M}\cdot\frac{z}{k}\right\} + \frac{\omega^2r^2}{2(1+\lambda)^2} \quad . \quad (12).$$

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The last term, being independent of x, y, z, has no present interest. Then, collecting results from (10), (11), and (12), the whole potential, as far as material, is

$$\tfrac{3}{2}\frac{M}{k}\left[\frac{2}{3\lambda}\left\{3\mathfrak{Q}_1\left(\frac{r}{k}\right)-\frac{\lambda\omega^2k^2r}{(1+\lambda)M}\right\}\frac{z}{k}\right.$$
$$-\frac{x^2}{k^2(\nu_0^2-1/\kappa^2)}\left\{\mathsf{A}_1^1+\frac{\nu_0^2-1/\kappa^2}{3\lambda}\frac{k^3}{r^3}\left[1+\frac{3(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2}+\frac{5(5+2\kappa^2+\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}\cdots\right]\right\}$$
$$-\frac{y^2}{k^2(\nu_0^2-1)}\left\{\mathfrak{A}_1^1+\frac{\nu_0^2-1}{3\lambda}\frac{k^3}{r^3}\left[1+\frac{3(3\kappa^2+1)}{14\kappa^2}\frac{k^2}{r^2}+\frac{5(5\kappa^4+2\kappa^2+1)}{56\kappa^4}\frac{k^4}{r^4}\cdots\right]-\frac{\omega^2k^3}{3M}(\nu_0^2-1)\right\}$$
$$\left.-\frac{z^2}{k^2\nu_0^2}\left\{\mathfrak{A}_1-\frac{2\nu_0^2}{3\lambda}\frac{k^3}{r^3}\left[1+\frac{3(1+\kappa^2)}{7\kappa^2}\frac{k^2}{r^2}+\frac{5(3+2\kappa^2+3\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}\cdots\right]-\frac{\omega^2k^3}{3M}\nu_0^2\right\}\right].$$

The condition that the figure of equilibrium should be the ellipsoid of reference is that this potential when equated to a constant should reproduce the equation to the ellipsoid. The coefficient of z must therefore vanish, and the three coefficients written inside { } must be equal to one another. These conditions give the angular velocity and equations for determining the figure, but as the subject will be reconsidered from a different point of view hereafter, I do not pursue the investigation here.

At present it need only be noted that the coefficient of z vanishes, and that the three coefficients are equal to one another. It is clear then that the potential U of the system, as rendered statical, may be written

$$U=-\frac{3M}{2k}\left\{\mathsf{A}_1^1+\frac{\nu_0^2-1/\kappa^2}{3\lambda}\frac{k^3}{r^3}\left[1+\frac{3(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2}+\frac{5(5+2\kappa^2+\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}\cdots\right]\right\}$$
$$\times\left\{\frac{x^2}{k^2(\nu_0^2-1/\kappa^2)}+\frac{y^2}{k^2(\nu_0^2-1)}+\frac{z^2}{k^2\nu_0^2}\right\}.$$

Now gravity g at the surface of the ellipsoid is $-dU/dn$, where n is the outward normal to the ellipsoid.

Hence

$$g=-\left[\frac{px}{k^2(\nu_0^2-1/\kappa^2)}\frac{\partial U}{\partial x}+\frac{py}{k^2(\nu_0^2-1)}\frac{\partial U}{\partial y}+\frac{pz}{k^2\nu_0^2}\frac{\partial U}{\partial z}\right].$$

Now

$$\frac{1}{p^2}=\frac{x^2}{k^4(\nu_0^2-1/\kappa^2)^2}+\frac{y^2}{k^4(\nu_0^2-1)^2}+\frac{z^2}{k^4\nu_0^4},$$

and in our alternative notation

$$\nu_0^2-\frac{1}{\kappa^2}=\frac{\cos^2\gamma}{\sin^2\beta}.$$

Therefore

$$g=\frac{3M}{pk}\left\{\mathsf{A}_1^1+\frac{k^3\cos^2\gamma}{3\lambda r^3\sin^2\beta}\left[1+\frac{3(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2}+\frac{5(5+2\kappa^2+\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}+\cdots\right]\right\}\quad.\quad(13).$$

As already remarked, we shall put $M = \frac{4}{3}\pi\rho a^3 \frac{\lambda}{1+\lambda}$; also, since the three axes of the ellipsoid are $k \cos\gamma \operatorname{cosec}\beta$, $k \cos\beta \operatorname{cosec}\beta$, $k \operatorname{cosec}\beta$, we have

$$\frac{k^3 \cos\beta \cos\gamma}{\sin^3\beta} = \frac{\lambda a^3}{1+\lambda}.$$

Hence

$$\frac{k^3 \cos^2\gamma}{3\lambda r^3 \sin^2\beta} = \frac{a^3 \cos\gamma \tan\beta}{3(1+\lambda)r^3},$$

and the coefficient of the series in the expression for g does not become infinite when λ vanishes; however, it is perhaps more convenient to leave it in the form as written above.

This expression for gravity is the result required, but it is to be noted that it is determined on the hypothesis that the distant body is a particle or sphere instead of being an ellipsoid.

The development ceases with terms of the seventh order, and the harmonic terms of third and higher orders have been neglected. Now the harmonic deformation of Roche's ellipsoid of the third order of harmonics is of order k^4/r^4 in inverse powers of r. This deformation is treated as surface density. If we were to proceed to closer approximation, we should have to take account of the square of the thickness of the layer; such terms would be of order k^8/r^8. Since, then, we are avowedly neglecting terms of this order, it is no use to carry the development higher than terms of the seventh order.

§ 6. *Form of the Expression for the Gravitational Lost Energy of the System.*

The system consists of two ellipsoids, say e and E, with their longest axes co-linear, and each of them is distorted by deformations expressible by ellipsoidal harmonics of orders higher than the second. To the order of approximation to be adopted these deformations may be replaced by layers of surface density, which may be denoted by l and L respectively.

The lost energy of the system may be represented symbolically by

$$V = \tfrac{1}{2}(e+l)^2 + \tfrac{1}{2}(E+L)^2 + (e+l)(E+L).$$

Let s, S denote two spheres of masses equal to e and E and concentric with them respectively.

Then the whole may be written

$$V = \tfrac{1}{2}ee + \tfrac{1}{2}EE + eE + \tfrac{1}{2}ll + \tfrac{1}{2}LL + (e+S)\,l + (E+s)\,L$$
$$+ (e-s)\,L + (E-S)\,l + lL.$$

In the term Sl I divide S into two parts, namely S_1, which is to contain all the terms in the potential of S at the surface of e excepting terms expressible by ellipsoidal harmonics of the second order with respect to the ellipsoid e; and S_2, which is to contain the omitted terms of the second order. Similarly, the term sL is to be divided into s_1L and s_2L.

Take the centre of e as origin of co-ordinates x, y, z with the z axis passing through the centre of E, the y axis coincident with the mean axis of e and the x axis coincident with the least axis of e.

Since l is expressible by harmonics higher than the second order, and since y^2+z^2 is expressible by harmonics of orders 0 and 2, it follows that the moment of inertia of the layer l about the axis is zero. If therefore ω is the angular velocity of the system, a contribution to the lost energy of the system which may be written symbolically $[\frac{1}{2}\omega^2(y^2+z^2)]\,l$ is zero.

It follows therefore that we may write

$$(e+S)\,l = [e+S_2+\tfrac{1}{2}\omega^2(y^2+z^2)]\,l+S_1l.$$

Similarly, if the ellipsoid E be referred to a parallel co-ordinate system X, Y, Z through its centre, and such that

$$x = X, \quad y = Y, \quad z = Z+r,$$

so that r is the distance between the two origins, we have

$$(E+s)\,L = [E+s_2+\tfrac{1}{2}\omega^2(Y^2+Z^2)]\,L+s_1L.$$

The problem is already so complicated that it will be convenient to omit certain small terms in the expression for the lost energy, which it would be very troublesome to evaluate.

The term $(e-s)\,L$ represents the mutual energy of the departure from sphericity of e with the layer of surface density L on E. This term is clearly very small and will be omitted. Similarly $(E-S)\,l$ will be neglected. It will appear from the results below that these terms are at least of the seventh order in powers of $1/r$. *À fortiori* lL, which is at least of the eighth order, will be omitted.

The whole expression for V will now be divided into several portions.

Let $(eE)_1$ be that portion of eE in which each ellipsoid may be replaced by a particle; it is, in fact, the product of the masses of e and E divided by r.

Let $(eE)_2$ be the rest of eE.

Let (vv) denote that portion of V in which the larger body E may be replaced by a sphere; then

$$(vv) = \tfrac{1}{2}ee+\tfrac{1}{2}ll+[e+S_2+\tfrac{1}{2}\omega^2(y^2+z^2)]\,l+S_1l.$$

Similarly, let

$$(VV) = \tfrac{1}{2}EE+\tfrac{1}{2}LL+[E+s_2+\tfrac{1}{2}\omega^2(Y^2+Z^2)]\,L+s_1L.$$

Then $V = (eE)_1+(eE)_2+(vv)+(VV)+$ neglected terms.

For ROCHE's problem, when the second body is a particle, V reduces to $(eE)_1+(vv)$, but in the modified form of the problem which I am going to solve the whole expression is required.

The evaluation of (eE) is so complicated that I devote a special section to it.

§ 7. *The Mutual Energy of Two Ellipsoids.**

The semi-axes of the ellipsoid of mass e are to be denoted a, b, c, and the corresponding notation for the other ellipsoid is E, A, B, C. The distance between the centres of e and E is r; the axes c of e and C of E are in the same straight line, while a, A and b, B are respectively parallel. For brevity I imagine the densities of the ellipsoids to be unity.

If the external potential of e be U, and if $d\Omega$ be an element of volume or of mass of E, the lost energy to be evaluated is

$$(eE) = \int U d\Omega,$$

integrated throughout the ellipsoid E.

Let us suppose provisionally that the co-ordinates of the centres e and E are x, y, z and X, Y, Z, and let ξ, η, ζ be the co-ordinates of the element $d\Omega$; the axes being respectively parallel to a, b, c or A, B, C with arbitrary origin.

If $R^2 = (\xi-x)^2+(\eta-y)^2+(\zeta-z)^2$, it is well known that the potential U of e at the point ξ, η, ζ is given by

$$U = e\sum_0^\infty \frac{3}{(2n+1)(2n+3)2n!}\left(a^2\frac{\partial^2}{\partial\xi^2}+b^2\frac{\partial^2}{\partial\eta^2}+c^2\frac{\partial^2}{\partial\zeta^2}\right)^n\frac{1}{R}.$$

Since $\frac{1}{R}$ satisfies LAPLACE's equation, we may eliminate $\frac{\partial^2}{\partial\zeta^2}$, and observing that $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$ are the same as $\frac{\partial^2}{\partial\xi^2}$, $\frac{\partial^2}{\partial\eta^2}$ respectively, we have

$$a^2\frac{\partial^2}{\partial\xi^2}+b^2\frac{\partial^2}{\partial\eta^2}+c^2\frac{\partial^2}{\partial\zeta^2} = -(c^2-a^2)\frac{\partial^2}{\partial\xi^2}-(c^2-b^2)\frac{\partial^2}{\partial\eta^2}$$

$$= -k^2\left[\frac{1}{\kappa^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right],$$

It follows therefore that

$$U = e\sum_0^\infty \frac{(-)^n 3}{(2n+1)(2n+3)2n!}k^{2n}\left(\frac{1}{\kappa^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^n\frac{1}{R}.$$

Since the operator is independent of ξ, η, ζ, we have

$$(eE) = e\sum_0^\infty \frac{(-)^n 3}{(2n+1)(2n+3)2n!}k^{2n}\left(\frac{1}{\kappa^2}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^n\cdot\int\frac{d\Omega}{R}.$$

* The results of this section were arrived at originally by a longer method. I have to thank one of the referees for showing me the following procedure.—*February* 26, 1906.

But $\int \frac{d\Omega}{R}$ is the potential of the ellipsoid E at the centre of the ellipsoid e, and by an exactly parallel transformation

$$\int \frac{d\Omega}{R} = E \sum_0^\infty \frac{(-)^n 3}{(2n+1)(2n+3)\,2n!} K^{2n} \left(\frac{1}{\mathrm{K}^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^n \frac{1}{\rho},$$

where $\rho^2 = (x-X)^2 + (y-Y)^2 + (z-Z)^2$.

Since our co-ordinate axes have a perfectly arbitrary origin, we may at once put $X = 0$, $Y = 0$, $Z = r$, $z = 0$, and after effecting the several differentiations put $x = 0$, $y = 0$.

It follows that, on putting $x = 0$, $y = 0$, after differentiation and writing $\rho^2 = x^2+y^2+r^2$,

$$(eE) = eE \sum_0^\infty \frac{(-)^n 3}{(2n+1)(2n+3)2n!} k^{2n} \left(\frac{1}{\kappa^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^n \sum_0^\infty \frac{(-)^i 3}{(2i+1)(2i+3)2i!} K^{2i} \left(\frac{1}{\mathrm{K}^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^i \frac{1}{\rho}.$$

If we denote the operator $\frac{1}{\kappa^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ by d^2, and the operator $\frac{1}{\mathrm{K}^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ by D^2, we have

$$(eE) = eE\left[1 - \frac{1}{2.5}k^2\mathrm{d}^2 + \frac{1}{2^3.5.7}k^4\mathrm{d}^4 - \frac{1}{2^4.3^3.5.7}k^6\mathrm{d}^6 \ldots\right]$$
$$\left[1 - \frac{1}{2.5}K^2\mathrm{D}^2 + \frac{1}{2^3.5.7}K^4\mathrm{D}^4 - \frac{1}{2^4.3^3.5.7}K^6\mathrm{D}^6 \ldots\right]\frac{1}{\rho}$$
$$= eE\left[1 - \frac{1}{2.5}(k^2\mathrm{d}^2 + K^2\mathrm{D}^2) + \frac{1}{2^3.5.7}(k^4\mathrm{d}^4 + K^4\mathrm{D}^4) + \frac{1}{2^2.5^2}k^2K^2\mathrm{d}^2\mathrm{D}^2\right.$$
$$\left. - \frac{1}{2^4.3^3.5.7}(k^6\mathrm{d}^6 + K^6\mathrm{D}^6) - \frac{1}{2^4.5^2.7}(k^2K^4\mathrm{d}^2\mathrm{D}^4 + k^4K^2\mathrm{d}^4\mathrm{D}^2)\ldots\right]\frac{1}{\rho}.$$

On effecting the several differentiations, and putting $x = 0$, $y = 0$, we find

$$\mathrm{d}^2\frac{1}{\rho} = -\frac{1}{r^3}\left(\frac{1}{\kappa^2}+1\right); \quad \mathrm{d}^4\frac{1}{\rho} = \frac{3}{r^5}\left(\frac{3}{\kappa^4}+\frac{2}{\kappa^2}+3\right); \quad \mathrm{D}^2\mathrm{d}^2\frac{1}{\rho} = \frac{3}{r^5}\left(\frac{3}{\kappa^2\mathrm{K}^2}+\frac{1}{\kappa^2}+\frac{1}{\mathrm{K}^2}+3\right);$$

$$\mathrm{d}^6\frac{1}{\rho} = -\frac{3^2.5}{r^7}\left(\frac{5}{\kappa^6}+\frac{3}{\kappa^4}+\frac{3}{\kappa^2}+5\right); \quad \mathrm{d}^4\mathrm{D}^2\frac{1}{\rho} = -\frac{3^2.5}{r^7}\left(\frac{5}{\kappa^4\mathrm{K}^2}+\frac{1}{\kappa^4}+\frac{2}{\kappa^2\mathrm{K}^2}+\frac{2}{\kappa^2}+\frac{1}{\mathrm{K}^2}+5\right);$$

and the remaining functions may be found by appropriate changes of small and large letters.

If now we again use ρ to denote the density of the spheroids, and revert to the notation employed elsewhere, namely,

$$e = \tfrac{4}{3}\pi\rho \mathrm{a}^3 \frac{\lambda}{1+\lambda}, \quad E = \tfrac{4}{3}\pi\rho \mathrm{a}^3 \frac{1}{1+\lambda},$$

we find

$$(eE) = (\tfrac{4}{3}\pi\rho a^3)^2 \frac{\lambda}{(1+\lambda)^2}\Big\{1 + \frac{1}{2.5r^3}\Big[k^2\Big(\frac{1}{\kappa^2}+1\Big) + K^2\Big(\frac{1}{K^2}+1\Big)\Big]$$

$$+ \frac{3}{2^3.5.7r^5}\Big[k^4\Big(\frac{3}{\kappa^4}+\frac{2}{\kappa^2}+3\Big) + K^4\Big(\frac{3}{K^4}+\frac{2}{K^2}+3\Big)\Big]$$

$$+ \frac{3}{2^2.5^2r^5}\Big[k^2K^2\Big(\frac{3}{\kappa^2K^2}+\frac{1}{\kappa^2}+\frac{1}{K^2}+3\Big)\Big]$$

$$+ \frac{1}{2^4.3.7r^7}\Big[k^6\Big(\frac{5}{\kappa^6}+\frac{3}{\kappa^4}+\frac{3}{\kappa^2}+5\Big) + K^6\Big(\frac{5}{K^6}+\frac{3}{K^4}+\frac{3}{K^2}+5\Big)\Big]$$

$$+ \frac{9}{2^4.5.7r^7}\Big[k^2K^4\Big(\frac{5}{\kappa^2K^4}+\frac{1}{K^4}+\frac{2}{\kappa^2K^2}+\frac{2}{K^2}+\frac{1}{\kappa^2}+5\Big)$$

$$+ k^4K^2\Big(\frac{5}{\kappa^4K^2}+\frac{1}{\kappa^4}+\frac{2}{\kappa^2K^2}+\frac{2}{\kappa^2}+\frac{1}{K^2}+5\Big)\Big]\Big\} \quad (14).$$

The first term in this expression is that which was called above $(eE)_1$, and the rest constitutes $(eE)_2$.

If the body E were a sphere, the only portions of (14) which would remain would be the parts of the expression independent of K.

With the object of effecting certain differentiations hereafter, it is desirable that the formula for (eE) should be expressed in terms of the semi-axes a, b, c and A, B, C.

In accordance with the notation used elsewhere, we have

$$a = \frac{k\cos\gamma}{\sin\beta}, \quad b = \frac{k\cos\beta}{\sin\beta}, \quad c = \frac{k}{\sin\beta}, \quad \text{where } \sin\beta = \kappa\sin\gamma,$$

$$A = \frac{K\cos\Gamma}{\sin B}, \quad B = \frac{K\cos B}{\sin B}, \quad C = \frac{K}{\sin B}, \quad \text{where } \sin B = K\sin\Gamma.*$$

The result of the translation into this other notation is as follows:—

$$(eE) = (\tfrac{4}{3}\pi\rho a^3)^2 \frac{\lambda}{(1+\lambda)^2}\Big\{1 + \frac{1}{2.5r^3}[2c^2-a^2-b^2+\text{same in } A, B, C]$$

$$+ \frac{3}{2^3.5.7r^5}[3(a^4+b^4)+8c^4-8c^2(a^2+b^2)+2a^2b^2+\text{same in } A, B, C]$$

$$+ \frac{3}{2^2.5^2r^5}[2(A^2a^2+B^2b^2+C^2c^2)+(A^2+B^2+C^2)(a^2+b^2+c^2)$$
$$-5C^2(a^2+b^2)-5c^2(A^2+B^2)+5C^2c^2]$$

$$+ \frac{1}{2^4.3.7r^7}[16c^6-5(a^6+b^6)-24c^4(a^2+b^2)+18c^2(a^4+b^4)$$
$$-3a^2b^2(a^2+b^2)+12a^2b^2c^2+\text{same in } A, B, C]$$

$$+ \frac{9}{2^4.5.7r^7}[-A^4(5a^2+b^2-6c^2)-B^4(a^2+5b^2-6c^2)$$
$$-8C^4(a^2+b^2-2c^2)+4B^2C^2(a^2+3b^2-4c^2)$$
$$+4C^2A^2(3a^2+b^2-4c^2)-2A^2B^2(a^2+b^2+2c^2)$$
$$+\text{same with small and large letters interchanged}]\Big\} \quad \ldots \ldots \quad (15).$$

* The fact that capital β is nearly the same as B must be pardoned; it cannot, I think, cause any confusion.

§ 8. *Remaining Terms in the Expression for the Lost Energy.*

If e be the mass of an ellipsoid of semi-axes a, b, c, the lost energy of its concentration is

$$\tfrac{1}{2}(ee) = \tfrac{3}{10}e^2\psi,$$

where

$$\psi = \int_0^\infty \frac{du}{(u+a^2)^{1/2}(u+b^2)^{1/2}(u+c^2)^{1/2}}.$$

In the present case

$$e = \tfrac{4}{3}\pi\rho \mathrm{a}^3 \frac{\lambda}{1+\lambda},$$

and

$$\psi = \frac{2}{k}\int_{\nu_0}^\infty \frac{d\nu}{(\nu^2-1)^{1/2}(\nu^2-1/\kappa^2)^{1/2}},$$

where

$$\nu_0 = \frac{1}{\kappa \sin\gamma} = \frac{1}{\sin\beta}.$$

Thus

$$\tfrac{1}{2}(ee) = \tfrac{3}{10}(\tfrac{4}{3}\pi\rho \mathrm{a}^3)^2 \frac{\lambda^2}{(1+\lambda)^2}\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

By symmetry

$$\tfrac{1}{2}(EE) = \tfrac{3}{10}(\tfrac{4}{3}\pi\rho \mathrm{a}^3)^2 \frac{1}{(1+\lambda)^2}\Psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

where

$$\Psi = \frac{2}{K}\int_{\mathrm{N}_0}^\infty \frac{d\mathrm{N}}{(\mathrm{N}^2-1)^{1/2}(\mathrm{N}^2-1/\mathrm{K}^2)^{1/2}},$$

and

$$\mathrm{N}_0 = \frac{1}{\mathrm{K}\sin\Gamma} = \frac{1}{\sin\mathrm{B}}.$$

The lost energy (lS_1) is the potential of a particle S, equal in mass to E placed at the centre of E, with the omission of terms of the second order of harmonics, multiplied by the density of the layer l and integrated over the surface of e. This is the same as the potential of the layer l, with the omission of harmonic terms of the second order (and there are none such) at the centre of E multiplied by the mass of E.

A typical term in the surface density representing the layer l is, say,

$$f_i^s \mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi).$$

The external potential corresponding to such a term, at the point ν, μ, ϕ, is by (51) of "Harmonics,"

$$\frac{3}{k}\left(\tfrac{4}{3}\pi\rho \mathrm{a}^3 \frac{\lambda}{1+\lambda}\right) f_i^s \mathfrak{Q}_i^s(\nu)\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi).$$

The co-ordinates of the centre of E are $\nu = \dfrac{r}{k}$, $\mu = 1$, $\phi = \tfrac{1}{2}\pi$; and the mass of E is $\tfrac{4}{3}\pi\rho \mathrm{a}^3/(1+\lambda)$.

Hence the contribution to (lS_1) corresponding to this term is

$$\frac{3}{k}\left(\tfrac{4}{3}\pi\rho a^3\right)^2 \frac{\lambda}{(1+\lambda)^2} f_i^s \mathfrak{Q}_i^s(r/k)\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^s(1)\,\mathfrak{C}_i^s(\tfrac{1}{2}\pi).$$

Now $\mathfrak{P}_i^s(1)\,\mathfrak{C}_i^s(\frac{1}{2}\pi)$ vanishes for all harmonics except cosine-harmonics of even rank. Therefore

$$(lS_1) = \frac{3}{k}\left(\tfrac{4}{3}\pi\rho a^3\right)^2 \frac{\lambda}{(1+\lambda)^2} \Sigma f_i^s \mathfrak{Q}_i^s(r/k)\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^s(1) \begin{Bmatrix} \mathfrak{C}_i^s \\ \mathsf{C}_i^s \end{Bmatrix} (\tfrac{1}{2}\pi) \begin{Bmatrix} i \text{ even} \\ i \text{ odd} \end{Bmatrix} \quad . \quad . \quad (18),$$

the summation being for all values of i greater than 2, and for all even values of s.

The lost energy (Ls_1) is expressible by the similar function for the other ellipsoid, but I have not adopted a specific notation for the ellipsoidal harmonics for the ellipsoid E, and therefore cannot write down the result.

The lost energy $(e+S_2+\frac{1}{2}\omega^2(y^2+z^2))l$ is the potential of the ellipsoid e, together with the potential of S in as far as it involves harmonics of the second order, and a rotation potential, multiplied by the density of the layer l, and integrated over the surface of e. That is to say it is the potential of gravity on the ellipsoid e integrated throughout the layer l, which it is not permissible to regard as surface density.

If the thickness of the layer be ζ, and if $d\zeta$ be a slice of that part of the layer which is erected normally on an element $d\sigma$ of the surface e, then $\rho\, d\zeta\, d\sigma$ is an element of mass of the layer. The potential of gravity is $-g\zeta$. Hence the lost energy is

$$-\rho \iint_0^\zeta g\zeta\, d\zeta\, d\sigma = -\tfrac{1}{2}\rho \int g\zeta^2\, d\sigma.$$

Accordingly the lost energy is equal and opposite to the work done in raising the layer, considered as surface density, through half its thickness, against gravity.

We may take as a typical term

$$\zeta = pf_i^s \mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi),$$

and we have shown in (13) that

$$g = \frac{3}{pk}\tfrac{4}{3}\pi\rho a^3 \frac{\lambda}{1+\lambda}\left\{\mathsf{A}_1{}^1 + \frac{k^3}{3\lambda r^3}\frac{\cos^2\gamma}{\cos^2\beta}\left[1+\frac{3(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2}+\frac{5(5+2\kappa^2+\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}\cdots\right]\right\}.$$

It should be noted that this expression for gravity takes no account of the change in the ellipticity of e which is due to the fact that E is an ellipsoid and not a sphere. The error introduced thus is however outside the limits of accuracy which have been adopted.

Accordingly this portion of the lost energy is

$$-\frac{3}{2k}\tfrac{4}{3}\pi\rho^2 a^3 \frac{\lambda}{1+\lambda}(f_i^s)^2\left\{\mathsf{A}_1{}^1+\text{series}\right\}\int\left[\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)\right]^2 p\, d\sigma.$$

Now in § 5 we defined

$$\mathfrak{T}_i^s = \frac{(2i+1)}{4\pi a^3\lambda/(1+\lambda)}\int [\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi)]^2 p\,d\sigma.$$

Hence

$$(e+S_2+\tfrac{1}{2}\omega^2(y^2+z^2))\,l = -\frac{9}{2k}(\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda^2}{(1+\lambda)^2}\Sigma\frac{(f_i^s)^2}{2i+1}\{\mathsf{A}_1{}^1+\text{series}\}\,\mathfrak{T}_i^s \quad . \quad (19),$$

the summation being for all harmonics.

In determining the lost energy $\frac{1}{2}(ll)$ we may treat the layer as surface density. A typical term in the surface density is $\rho p f_i^s \mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi)$, and the surface value of its potential is

$$\frac{3}{k}(\tfrac{4}{3}\pi\rho a^3)\frac{\lambda}{1+\lambda}f_i^s\mathfrak{P}_i^s(\nu_0)\mathfrak{Q}_i^s(\nu_0)\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi).$$

Then, since

$$\mathfrak{A}_i^s = \mathfrak{P}_i^s(\nu_0)\mathfrak{Q}_i^s(\nu_0),$$

a typical term of $\frac{1}{2}(ll)$ is

$$\frac{3\rho}{2k}(\tfrac{4}{3}\pi\rho a^3)\frac{\lambda}{1+\lambda}(f_i^s)^2\mathfrak{A}_i^s\int [\mathfrak{P}_i^s(\mu)\mathfrak{C}_i^s(\phi)]^2 p\,d\sigma.$$

Thus

$$\tfrac{1}{2}(ll) = \frac{9}{2k}(\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda^2}{(1+\lambda)^2}\Sigma\frac{(f_i^s)^2}{2i+1}\mathfrak{A}_i^s\mathfrak{T}_i^s \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

the summation being made for all harmonics.

The value of $\frac{1}{2}(LL)$ may be written down by symmetry.

§ 9. *Final Expression for the Lost Energy of the System.*

We have

$$V = (eE)_1 + (vv) + (VV) + (eE)_2.$$

The several parts are to be collected from (14) or (15), (16), (17), (18), (19), (20), and we have

$$(eE)_1 = (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2 r},$$

$$(vv) = (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\Big\{\tfrac{3}{10}\lambda\psi + \frac{3}{k}\Sigma f_i^s\mathfrak{Q}_i^s(r/k)\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^s(1)\,\mathfrak{C}_i^s(\tfrac{1}{2}\pi) \quad (i>2,\ s\text{ even})$$

$$+\tfrac{9}{2}\frac{\lambda}{k}\Sigma\frac{(f_i^s)^2\mathfrak{T}_i^s}{2i+1}\Big[\mathfrak{A}_i^s - \mathsf{A}_1{}^1 - \frac{k^3\cos^2\gamma}{3\lambda r^3\sin^2\beta}\Big(1+\frac{3(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2}+\frac{5(5+2\kappa^2+\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}\Big)\Big]\begin{matrix}(\text{all}\\ \text{harm.})\end{matrix}\Big\},$$

(VV)= symmetrical expression with $1/\lambda$ in place of λ.

$$(eE)_2 = (\tfrac{4}{3}\pi\rho a^3)^2 \frac{\lambda}{(1+\lambda)^2}\left\{\frac{1}{2.5r^3}\left[k^2\left(\frac{1}{\kappa^2}+1\right)+K^2\left(\frac{1}{K^2}+1\right)\right]\right.$$
$$+\frac{3}{2^3.5.7r^5}\left[k^4\left(\frac{3}{\kappa^4}+\frac{2}{\kappa^2}+3\right)+K^4\left(\frac{3}{K^4}+\frac{2}{K^2}+3\right)\right]$$
$$+\frac{3}{2^2.5^2r^5}k^2K^2\left(\frac{3}{\kappa^2K^2}+\frac{1}{K^2}+\frac{1}{\kappa^2}+3\right)$$
$$+\frac{1}{2^4.3.7r^7}\left[k^6\left(\frac{5}{\kappa^6}+\frac{3}{\kappa^4}+\frac{3}{\kappa^2}+5\right)+K^6\left(\frac{5}{K^6}+\frac{3}{K^4}+\frac{3}{K^2}+5\right)\right]$$
$$+\frac{9}{2^4.5.7r^7}\left[k^2K^4\left(\frac{5}{\kappa^2K^4}+\frac{1}{K^4}+\frac{2}{\kappa^2K^2}+\frac{2}{K^2}+\frac{1}{\kappa^2}+5\right)\right.$$
$$\left.\left.+k^4K^2\left(\frac{5}{K^2\kappa^4}+\frac{1}{\kappa^4}+\frac{2}{K^2\kappa^2}+\frac{2}{\kappa^2}+\frac{1}{K^2}+5\right)\right]\right\} \quad (21).$$

§ 10. *Determination of the Forms of the Ellipsoids.*

We have obtained in the last section the expression for V, the lost energy of the system.

The harmonic deformations of the ellipsoids being of orders higher than the second do not enter into the moment of inertia to the order of approximation adopted. Hence the moment of inertia about the axis of rotation, which passes through the centre of inertia of the system, and is parallel to the a and A axes of the ellipsoids, is given by

$$I = \tfrac{4}{3}\pi\rho a^3\left[\tfrac{1}{5}\frac{\lambda}{1+\lambda}(b^2+c^2)+\tfrac{1}{5}\frac{1}{1+\lambda}(B^2+C^2)+\frac{\lambda r^2}{(1+\lambda)^2}\right]. \quad . \quad . \quad . \quad (22).$$

If f denotes any one of the parameters by which the system is defined, the condition that the figures shall be in equilibrium is

$$\frac{\partial V}{\partial f}+\tfrac{1}{2}\omega^2\frac{\partial I}{\partial f}=0.$$

The parameters defining the system may be taken as r, the distance between the two centres, $\cos\gamma$, $\cos\beta$ for the smaller ellipsoid and $\cos\Gamma$, $\cos \mathrm{B}$ for the larger one. Besides these we have the coefficients f_i^s, F_i^s of the harmonic inequalities of order i and rank s on the two ellipsoids.

For convenience write

$$\mathfrak{a} = \cos\gamma, \quad \mathfrak{b} = \cos\beta.$$

These letters are chosen on account of the association of $\cos\gamma$, $\cos\beta$ with the semi-axes a, b of the smaller ellipsoid e. It is unnecessary to adopt a corresponding notation for the ellipsoid E, because, when the problem is solved as regards e, it affords the solution for E by symmetry.

Since

$$k^3\cos\beta\cos\gamma\,\mathrm{cosec}^3\,\beta = \lambda a^3/(1+\lambda),$$

we have

$$\frac{k}{\mathrm{a}} = \left(\frac{\lambda}{1+\lambda}\right)^{1/3} \frac{(1-\mathfrak{b}^2)^{1/2}}{(\mathfrak{a}\mathfrak{b})^{1/3}}.$$

Hence

$$\frac{a}{\mathrm{a}} = \frac{k}{\mathrm{a}}\frac{\cos\gamma}{\sin\beta} = \left(\frac{\lambda}{1+\lambda}\right)^{1/3}\frac{\mathfrak{a}^{2/3}}{\mathfrak{b}^{1/3}}, \quad \frac{b}{\mathrm{a}} = \frac{k}{\mathrm{a}}\frac{\cos\beta}{\sin\beta} = \left(\frac{\lambda}{1+\lambda}\right)^{1/3}\frac{\mathfrak{b}^{2/3}}{\mathfrak{a}^{1/3}}, \quad \frac{c}{\mathrm{a}} = \frac{k}{\mathrm{a}\sin\beta} = \left(\frac{\lambda}{1+\lambda}\right)^{1/3}\frac{1}{(\mathfrak{a}\mathfrak{b})^{1/3}}.$$

Therefore

$$3\frac{da}{a} = 2\frac{d\mathfrak{a}}{\mathfrak{a}} - \frac{d\mathfrak{b}}{\mathfrak{b}}, \quad 3\frac{db}{b} = -\frac{d\mathfrak{a}}{\mathfrak{a}} + 2\frac{d\mathfrak{b}}{\mathfrak{b}}, \quad 3\frac{dc}{c} = -\frac{d\mathfrak{a}}{\mathfrak{a}} - \frac{d\mathfrak{b}}{\mathfrak{b}}.$$

Therefore

$$3\mathfrak{a}\frac{\partial}{\partial\mathfrak{a}} = 2a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c}, \quad 3\mathfrak{b}\frac{\partial}{\partial\mathfrak{b}} = -a\frac{\partial}{\partial a} + 2b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} \quad . \quad . \quad . \quad (23).$$

These enable us to differentiate, with respect to $\mathfrak{a}$, $\mathfrak{b}$, functions expressed in terms of a, b, c; the parameters r, f_i^s always occur explicitly.

The equation of condition for the parameter r is

$$\frac{\partial V}{\partial r} + \tfrac{1}{2}\omega^2\frac{\partial I}{\partial r} = 0.$$

On differentiating (22) we have

$$\tfrac{1}{2}\omega^2\frac{\partial I}{\partial r} = \tfrac{4}{3}\pi\rho\mathrm{a}^3\frac{\lambda\omega^2 r}{(1+\lambda)^2}.$$

In order to differentiate V we must take separately its several portions as defined in (21).

Now

$$-\frac{\partial}{\partial r}(eE)_1 = (\tfrac{4}{3}\pi\rho\mathrm{a}^3)^2\frac{\lambda}{(1+\lambda)^2 r^2}$$

$$-\frac{\partial}{\partial r}(vv) = (\tfrac{4}{3}\pi\rho\mathrm{a}^3)^2\frac{\lambda}{(1+\lambda)^2}\Bigg\{-\frac{3}{k}\Sigma f_i^s\mathfrak{P}_i^s(\nu_0)\mathfrak{P}_i^s(1)\mathfrak{C}_i^s(\tfrac{1}{2}\pi)\frac{d}{dr}\mathfrak{Q}_i^s\left(\frac{r}{k}\right) \quad (i>2,\ s \text{ even})$$
$$+\tfrac{9}{2}\frac{k^2}{r^4}\Sigma\frac{(f_i^s)^2\,\mathfrak{T}_i^s}{2i+1}\frac{\cos^2\gamma}{\sin^2\beta}\left[1+\frac{5(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2}+\frac{5(5+2\kappa^2+\kappa^4)}{56\kappa^4}\frac{k^4}{r^4}\ldots\right](\text{all harm.})\Bigg\}$$

$-\frac{\partial}{\partial r}(VV) =$ symmetrical expression for larger ellipsoid.

$$-\frac{\partial}{\partial r}(eE)_2 = (\tfrac{4}{3}\pi\rho\mathrm{a}^3)^2\frac{\lambda}{(1+\lambda)^2}\Bigg\{\frac{3}{2.5r^4}\left[k^2\left(\frac{1}{\kappa^2}+1\right)+K^2\left(\frac{1}{\mathrm{K}^2}+1\right)\right]$$
$$+\frac{3}{2^3.7r^6}\left[k^4\left(\frac{3}{\kappa^4}+\frac{2}{\kappa^2}+3\right)+K^4\left(\frac{3}{\mathrm{K}^4}+\frac{2}{\mathrm{K}^2}+3\right)\right]$$
$$+\frac{3}{2^2.5r^6}k^2K^2\left(\frac{3}{\kappa^2\mathrm{K}^2}+\frac{1}{\kappa^2}+\frac{1}{\mathrm{K}^2}+3\right)$$
$$+\frac{1}{2^4.3r^8}\left[k^6\left(\frac{5}{\kappa^6}+\frac{3}{\kappa^4}+\frac{3}{\kappa^2}+5\right)+K^6\left(\frac{5}{\mathrm{K}^6}+\frac{3}{\mathrm{K}^4}+\frac{3}{\mathrm{K}^2}+5\right)\right]$$
$$+\frac{9}{2^4.5r^8}\Bigg[k^2K^4\left(\frac{5}{\kappa^2\mathrm{K}^4}+\frac{1}{\mathrm{K}^4}+\frac{2}{\kappa^2\mathrm{K}^2}+\frac{2}{\mathrm{K}^2}+\frac{1}{\kappa^2}+5\right)$$
$$+k^4K^2\left(\frac{5}{\mathrm{K}^2\kappa^4}+\frac{1}{\kappa^4}+\frac{2}{\kappa^2\mathrm{K}^2}+\frac{2}{\kappa^2}+\frac{1}{\mathrm{K}^2}+5\right)\Bigg]\Bigg\}.$$

The sum of these last four expressions is equal to $-\partial V/\partial r$, and therefore equal to $\frac{1}{2}\omega^2 \partial I/\partial r$.

Now let

$$\omega^2 r^3 = \tfrac{4}{3}\pi\rho a^3(1+\zeta),$$

so that ζ represents the correction to KEPLER'S law of periodic times on account of the ellipticity of the two bodies. Then we have

$$\begin{aligned}\zeta = {} & \frac{3}{10r^2}\left[k^2\left(\frac{1}{\kappa^2}+1\right)+K^2\left(\frac{1}{\mathrm{K}^2}+1\right)\right]+\frac{3}{56r^4}\left[k^4\left(\frac{3}{\kappa^4}+\frac{2}{\kappa^2}+3\right)+K^4\left(\frac{3}{\mathrm{K}^4}+\frac{2}{\mathrm{K}^2}+3\right)\right] \\ & +\frac{3}{20r^4}k^2K^2\left(\frac{3}{\kappa^2\mathrm{K}^2}+\frac{1}{\mathrm{K}^2}+\frac{1}{\kappa^2}+3\right)+\frac{1}{48r^6}\left[k^6\left(\frac{5}{\kappa^6}+\frac{3}{\kappa^4}+\frac{3}{\kappa^2}+5\right)+K^6\left(\frac{5}{\mathrm{K}^6}+\frac{3}{\mathrm{K}^4}+\frac{3}{\mathrm{K}^2}+5\right)\right] \\ & +\frac{9}{80r^6}\left[k^2K^4\left(\frac{5}{\kappa^2\mathrm{K}^4}+\frac{1}{\mathrm{K}^4}+\frac{2}{\mathrm{K}^2\kappa^2}+\frac{2}{\mathrm{K}^2}+\frac{1}{\kappa^2}+5\right)+k^4K^2\left(\frac{5}{\mathrm{K}^2\kappa^4}+\frac{1}{\kappa^4}+\frac{2}{\mathrm{K}^2\kappa^2}+\frac{2}{\kappa^2}+\frac{1}{\mathrm{K}^2}+5\right)\right] \\ & -\frac{3}{k}\Sigma f_i^s \mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^s(1)\,\mathfrak{C}_i^s(\tfrac{1}{2}\pi)\,r^2\frac{d}{dr}\mathfrak{Q}_i^s\left(\frac{r}{k}\right) \quad (i>2,\ s \text{ even}) \\ & +\tfrac{9}{2}\frac{k^2}{r^2}\Sigma\frac{(f_i^s)^2\,\mathfrak{T}_i^s}{2i+1}\frac{\cos^2\gamma}{\sin^2\beta}\left\{1+\frac{5(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2}+\ldots\right\} \quad \text{(all harmonics)} \quad . \; . \; . \; . \quad (24)\end{aligned}$$

When $\mathfrak{Q}_i^s(r/k)$ is developed in powers of $1/r$, its first term is one in $r^{-(i+1)}$; hence $r^2\dfrac{\partial}{\partial r}\mathfrak{Q}_i^s$ begins with r^{-i}. Now f_i^s will be determined from terms in the potential of the ellipsoid E of the i^{th} order of harmonics, and will therefore involve $r^{-(i+1)}$. Therefore in the series contained in the last term but one of ζ each term is of order $r^{-(2i+1)}$. Since the lowest value of i is 3, the term of lowest order in this series is one in r^{-7}, and as I shall not attempt to evaluate ζ beyond r^{-6} the whole of this series is negligible.

Again, since $(f_i^s)^2$ is of order r^{-2i-2}, and since r^{-2} occurs as a factor, each term is of order r^{-2i-4}. Thus the lowest term is of order r^{-10} and is negligible.*

It follows that the only sensible part of ζ arises from the portion of V denoted $(eE)_2$, and the last two terms of (24) may be erased.

We next consider the parameter f_i^s, and, since I does not involve it, the equation reduces to $\partial V/\partial f_i^s = 0$; or, since V only contains f_i^s in the part denoted (vv), it becomes $\partial(vv)/\partial f_i^s = 0$.

This gives, for $i>2$, s even

$$f_i^s = -\tfrac{1}{3}(2i+1)\frac{\mathfrak{Q}_i^s(r/k)\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^s(1)\,\mathfrak{C}_i^s(\tfrac{1}{2}\pi)}{\lambda\mathfrak{T}_i^s\left\{\mathfrak{A}_i^s-\mathsf{A}_1^1-\dfrac{k^3}{3\lambda r^3}\dfrac{\cos^2\gamma}{\sin^2\beta}\left[1+\dfrac{3(3+\kappa^2)}{14\kappa^2}\dfrac{k^2}{r^2}+\ldots\right]\right\}} \quad . \quad (25).$$

Since this formula contains λ in the denominator, it would appear at first sight as if

* It is proper to remark that the terms retained in ζ are really of higher orders than they appear to be. I recur to the neglected portions of ζ hereafter in § 23.

f_i^s became infinite when $\lambda = 0$. But this is not so, because when $\mathfrak{Q}_i^s(r/k)$ is developed the first term of the series is one in $(k/r)^{i+1}$; now $k^3 = \frac{\lambda}{1+\lambda}\, \mathrm{a}^3 \sin^3\beta \sec\beta \sec\gamma$, and therefore the formula for f_i^s involves the factor $\left(\frac{\lambda}{1+\lambda}\right)^{(i+1)/3} \cdot \frac{1}{\lambda}$ or $\left(\frac{\lambda}{1+\lambda}\right)^{(i-2)/3} \cdot \frac{1}{1+\lambda}$. We see then that f_i^s vanishes both when $\lambda = 0$ and $\lambda = \infty$.

This factor is a maximum when $\lambda = \frac{1}{3}(i-2)$. Therefore we should expect, *cæteris paribus*, the third harmonics to be most important when $\lambda = \frac{1}{3}$, the fourth when $\lambda = \frac{2}{3}$, the fifth when $\lambda = 1$, and the higher harmonics when λ is greater than unity. This prevision is partially fulfilled by the numerical results given below, but it was not to be expected that it should be exactly so, because the other conditions are not exactly the same in the solutions for various values of λ.

The formula shows, as stated above, that f_i^s is of order r^{-i-1}. The series in the denominator affects the result but slightly and might be omitted, except, perhaps, in the case of the third zonal harmonic. For all harmonics other than cosine-harmonics of even rank f_i^s is zero.

It is now possible to eliminate f_i^s from (vv) by substituting for it its value. These terms in (vv) become, in this way, equal to

$$(\tfrac{4}{3}\pi\rho \mathrm{a}^3)^2 \frac{\lambda}{(1+\lambda)^2} \left\{ -\frac{2i+1}{2k\lambda} \frac{\left[\mathfrak{Q}_i^s\left(\frac{r}{k}\right) \mathfrak{P}_i^s(\nu_0)\, \mathfrak{P}_i^s(1)\, \mathfrak{C}_i^s(\tfrac{1}{2}\pi)\right]^2}{\mathfrak{T}_i^s\,[\mathfrak{A}_i^s - \mathbf{A}_1{}^1 - \text{series}]} \right\} \quad . \quad . \quad (26).$$

When $i = 3$, this term is of order r^{-8}, and is negligible; hence we need no longer pay any attention to the inequalities on the ellipsoid. However, the formula (25) is important as rendering it possible to evaluate the inequalities.

Since for all inequalities, excepting cosine-harmonics of even rank, f_i^s only occurs in the energy function as a square, it is in these cases a principal co-ordinate, and $(\mathfrak{A}_i^s - \mathbf{A}_1{}^1 - \textit{series})$ is a coefficient of stability.

But the like is not true for the cosine-harmonics of even rank, because, when we consider, for example, the harmonics of the third order, we see that $\partial^2 V/df_3^s dr$ is of the fifth order and $\partial^2 V/\partial f_3^s \partial \mathfrak{a}$, $\partial^2 V/\partial f_3^s \partial \mathfrak{b}$ $(s = 0, 2)$ are of the fourth order.

It is clear that the inequalities on the ellipsoid E are determinable by symmetrical formulæ.

We must now turn to the equations of equilibrium for the parameters $\mathfrak{a}$ and $\mathfrak{b}$. Since differentiation with respect to these parameters is effected most conveniently by means of the formulæ (23), the portion of V called $(eE)_2$ should be written in the form (15). After effecting the differentiations it is, however, best to revert to the notation involving k, κ, γ, $\mathcal{K}$, K, Γ; but as an exception to the general rule as to notation, it is most convenient to retain the differentials of $k^2(1+1/\kappa^2)$ and of b^2+c^2 in the forms involving a, b, c. As the algebraic processes involved are rather long, I simply give the results, as follows:—

$$3\mathfrak{a}\frac{d}{d\mathfrak{a}}(b^2+c^2) = -2(b^2+c^2) \ldots \text{(i)},$$

$$3\mathfrak{b}\frac{d}{d\mathfrak{b}}(b^2+c^2) = 2(2b^2-c^2) \ldots \text{(i)}'.$$

$$3\mathfrak{a}\frac{d}{d\mathfrak{a}}k^2\left(\frac{1}{\kappa^2}+1\right) = 2(b^2-2a^2-2c^2) \ldots \text{(ii)},$$

$$3\mathfrak{b}\frac{d}{d\mathfrak{b}}k^2\left(\frac{1}{\kappa^2}+1\right) = 2(a^2-2b^2-2c^2) \ldots \text{(ii)}'.$$

The remaining results are expressed in terms of k, κ, γ, &c.

$$3\mathfrak{a}\frac{d}{d\mathfrak{a}}k^4\left(\frac{3}{\kappa^4}+\frac{2}{\kappa^2}+3\right) = \frac{4k^4}{\kappa^4\sin^2\gamma}\left[-3(3+\kappa^2)+(6+\kappa^2-3\kappa^4)\sin^2\gamma\right] \ldots \text{(iii)},$$

$$3\mathfrak{b}\frac{d}{d\mathfrak{b}} \quad \text{same} \quad = \frac{4k^4}{\kappa^4\sin^2\gamma}\left[-3(3\kappa^2+1)+(6\kappa^4+\kappa^2-3)\sin^2\gamma\right] \ldots \text{(iii)}'.$$

$$3\mathfrak{a}\frac{d}{d\mathfrak{a}}k^6\left(\frac{5}{\kappa^6}+\frac{3}{\kappa^4}+\frac{3}{\kappa^2}+5\right) = \frac{6k^6}{\kappa^6\sin^2\gamma}\left[-3(5+2\kappa^2+\kappa^4)+(10+3\kappa^2+0\kappa^4-5\kappa^6)\sin^2\gamma\right] \ldots \text{(iv)},$$

$$3\mathfrak{b}\frac{d}{d\mathfrak{b}} \quad \text{same} \quad = \frac{6k^6}{\kappa^6\sin^2\gamma}\left[-3(5\kappa^4+2\kappa^2+1)+(10\kappa^6+3\kappa^4+0\kappa^2-5)\sin^2\gamma\right] \ldots \text{(iv)}'.$$

$$3\mathfrak{a}\frac{d}{d\mathfrak{a}}k^2K^2\left(\frac{3}{\kappa^2\mathrm{K}^2}+\frac{1}{\mathrm{K}^2}+\frac{1}{\kappa^2}+3\right) = \frac{2k^2K^2}{\kappa^2\mathrm{K}^2\sin^2\gamma}\left[-(3+\mathrm{K}^2)(3-2\sin^2\gamma)-\kappa^2(1+3\mathrm{K}^2)\sin^2\gamma\right] \ldots \text{(v)},$$

$$3\mathfrak{b}\frac{d}{d\mathfrak{b}} \quad \text{same} \quad = \frac{2k^2K^2}{\kappa^2\mathrm{K}^2\sin^2\gamma}\left[-(3\mathrm{K}^2+1)(3-2\kappa^2\sin^2\gamma)-(\mathrm{K}^2+3)\sin^2\gamma\right] \ldots \text{(v)}'.$$

$$3\mathfrak{a}\frac{d}{d\mathfrak{a}}\left[k^2K^4\left(\frac{5}{\kappa^2\mathrm{K}^4}+\frac{1}{\mathrm{K}^4}+\frac{2}{\kappa^2\mathrm{K}^2}+\frac{2}{\mathrm{K}^2}+\frac{1}{\kappa^2}+5\right)\right.$$
$$\left.+\text{ same with small and large interchanged}\right] = \frac{2k^2K^4}{\kappa^2\mathrm{K}^4\sin^2\gamma}\left[-(5+2\mathrm{K}^2+\mathrm{K}^4)(3-2\sin^2\gamma)-\kappa^2(1+2\mathrm{K}^2+5\mathrm{K}^4)\sin^2\gamma\right]$$
$$+\frac{4k^4K^2}{\kappa^4\mathrm{K}^2\sin^2\gamma}\left[-3(5+\kappa^2)+(10+\kappa^2-\kappa^4)\sin^2\gamma\right.$$
$$\left.-\mathrm{K}^2\{3(1+\kappa^2)-(2+\kappa^2-5\kappa^4)\sin^2\gamma\}\right] \ldots \text{(vi)},$$

$$3\mathfrak{b}\frac{d}{d\mathfrak{b}} \quad \text{same} \quad = \frac{2k^2K^4}{\kappa^2\mathrm{K}^4\sin^2\gamma}\left[-(5+2\mathrm{K}^2+\mathrm{K}^4)\sin^2\gamma-(1+2\mathrm{K}^2+5\mathrm{K}^4)(3-2\kappa^2\sin^2\gamma)\right]$$
$$+\frac{4k^4K^2}{\kappa^4\mathrm{K}^2\sin^2\gamma}\left[-3(1+\kappa^2)+(2\kappa^4+\kappa^2-5)\sin^2\gamma\right.$$
$$\left.-\mathrm{K}^2\{3(5\kappa^2+1)-(10\kappa^4+\kappa^2-1)\sin^2\gamma\}\right] \ldots \text{(vi)}'.$$

On picking out the numerical coefficients of the several terms in $(eE)_2$ as given in (14) or (15), we see that

$$3\mathfrak{a}\frac{d}{d\mathfrak{a}}(eE)_2 = \frac{(\frac{4}{3}\pi\rho a^3)^2\lambda}{(1+\lambda)^2}\left\{\frac{1\,(\text{ii})}{2.5r^3}+\frac{3\,(\text{iii})}{2^3.5.7r^5}+\frac{3\,(\text{v})}{2^2.5^2r^5}+\frac{1\,(\text{iv})}{2^4.3.7r^7}+\frac{9\,(\text{vi})}{2^4.5.7r^7}\right\}.$$

$$3\mathfrak{b}\frac{d}{d\mathfrak{b}}(eE)_2 = \quad .\quad .\quad .\quad .\quad (\text{ii})'.\quad .\,(\text{iii})'.\quad .\,(\text{v})'.\quad .\,(\text{iv})'.\quad .\,(\text{vi})'.$$

Observe that $\dfrac{k^2}{\kappa^2\sin^2\gamma} = \dfrac{k^2}{\sin^2\beta} = c^2$, and write

$$\left.\begin{aligned}\tau &= \frac{3}{2^3.7r^2}\frac{\kappa^2\sin^2\gamma}{k^2}(\text{iii})+\frac{3}{2^2.5r^2}\frac{\kappa^2\sin^2\gamma}{k^2}(\text{v})+\frac{5}{2^4.3.7r^4}\frac{\kappa^2\sin^2\gamma}{k^2}(\text{iv})+\frac{9}{2^4.7r^4}\frac{\kappa^2\sin^2\gamma}{k^2}(\text{vi})\\ \sigma &= \quad .\quad .\quad .\quad .\,(\text{iii})'\,.\quad .\quad .\quad .\quad .\,(\text{v})'\quad .\quad .\quad .\quad .\quad .\,(\text{iv})'\,.\quad .\quad .\quad .\quad .\,(\text{vi})'\end{aligned}\right\}\quad (27).$$

Then we have

$$\left.\begin{aligned}3\mathfrak{a}\frac{d}{d\mathfrak{a}}(eE)_2 &= (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\left[\frac{1}{5r^3}(b^2-2a^2-2c^2)+\frac{1}{5r^3}\cdot\tau c^2\right]\\ 3\mathfrak{b}\frac{d}{d\mathfrak{b}}(eE)_2 &= (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\left[\frac{1}{5r^3}(a^2-2b^2-2c^2)+\frac{1}{5r^3}\cdot\sigma c^2\right]\end{aligned}\right\}\quad .\quad .\quad (28).$$

The terms in V denoted $(eE)_1$ and (VV) do not contain a, b, c, and their differentials with respect to $\mathfrak{a}$, $\mathfrak{b}$ are zero; also, after omission of the terms in f_i^s, (vv) is reduced to

$$(vv) = (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\cdot\tfrac{3}{10}\lambda\psi.$$

Hence

$$3\mathfrak{a}\frac{d}{d\mathfrak{a}}(vv) = (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\left[\tfrac{3}{10}\lambda\left(2a\frac{\partial\psi}{\partial a}-b\frac{\partial\psi}{\partial b}-c\frac{\partial\psi}{\partial c}\right)\right],$$

$$3\mathfrak{b}\frac{d}{d\mathfrak{b}}(vv) = (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\left[\tfrac{3}{10}\lambda\left(2b\frac{\partial\psi}{\partial b}-a\frac{\partial\psi}{\partial a}-c\frac{\partial\psi}{\partial c}\right)\right].$$

Now

$$a\frac{\partial\psi}{\partial a} = -\frac{2}{k}\mathsf{P}_1^1(\nu_0)\,\mathsf{Q}_1^1(\nu_0) = -\frac{2}{k}\mathsf{A}_1^1,$$

$$b\frac{\partial\psi}{\partial b} = -\frac{2}{k}\mathfrak{P}_1^1(\nu_0)\,\mathfrak{Q}_1^1(\nu_0) = -\frac{2}{k}\mathfrak{A}_1^1,$$

$$c\frac{\partial\psi}{\partial c} = -\frac{2}{k}\mathfrak{P}_1(\nu_0)\,\mathfrak{Q}_1(\nu_0) = -\frac{2}{k}\mathfrak{A}_1.$$

Since ψ is homogeneous of degree -1 in a, b, c, the sum of these three is equal to -1, so that

$$\mathsf{A}_1^1+\mathfrak{A}_1^1+\mathfrak{A}_1 = \tfrac{1}{2}k\psi.$$

Therefore

$$\left.\begin{aligned} 3\mathfrak{a}\frac{d}{d\mathfrak{a}}(vv) &= -(\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\cdot\frac{3\lambda}{5k}(3\mathbf{A}_1{}^1-\tfrac{1}{2}k\psi),\\ 3\mathfrak{b}\frac{d}{d\mathfrak{b}}(vv) &= -(\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\cdot\frac{3\lambda}{5k}(3\mathfrak{A}_1{}^1-\tfrac{1}{2}k\psi), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (29).$$

On adding together (28) and (29), we find

$$3\mathfrak{a}\frac{dV}{d\mathfrak{a}} = (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\left[-\frac{3\lambda}{5k}(3\mathbf{A}_1{}^1-\tfrac{1}{2}k\psi)+\frac{1}{5r^3}(b^2-2a^2-2c^2+\tau c^2)\right],$$

$$3\mathfrak{b}\frac{dV}{d\mathfrak{b}} = (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\left[-\frac{3\lambda}{5k}(3\mathfrak{A}_1{}^1-\tfrac{1}{2}k\psi)+\frac{1}{5r^3}(a^2-2b^2-2c^2+\sigma c^2)\right].$$

By means of (i) and (i)′ we find the differentials of the moment of inertia I; ıey are

$$3\mathfrak{a}\frac{dI}{d\mathfrak{a}} = -\tfrac{4}{3}\pi\rho a^3\frac{\lambda}{1+\lambda}\cdot\tfrac{2}{5}(b^2+c^2),$$

$$3\mathfrak{b}\frac{dI}{d\mathfrak{b}} = \tfrac{4}{3}\pi\rho a^3\frac{\lambda}{1+\lambda}\cdot\tfrac{2}{5}(2b^2-c^2).$$

Then, since $\tfrac{1}{2}\omega^2 = \tfrac{4}{3}\pi\rho a^3\cdot\frac{1}{2r^3}(1+\zeta)$,

$$\tfrac{3}{2}\omega^2\mathfrak{a}\frac{dI}{d\mathfrak{a}} = -(\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\cdot\frac{1}{5r^3}(1+\lambda)(1+\zeta)(b^2+c^2),$$

$$\tfrac{3}{2}\omega^2\mathfrak{b}\frac{dI}{d\mathfrak{b}} = (\tfrac{4}{3}\pi\rho a^3)^2\frac{\lambda}{(1+\lambda)^2}\cdot\frac{1}{5r^3}(1+\lambda)(1+\zeta)(2b^2-c^2).$$

Now the equations for equilibrium for the parameters $\mathfrak{a}$ and $\mathfrak{b}$ are

$$\frac{dV}{d\mathfrak{a}}+\tfrac{1}{2}\omega^2\frac{dI}{d\mathfrak{a}} = 0, \quad \frac{dV}{d\mathfrak{b}}+\tfrac{1}{2}\omega^2\frac{dI}{d\mathfrak{b}} = 0.$$

Therefore

$$\left.\begin{aligned} -3\lambda(3\mathbf{A}_1{}^1-\tfrac{1}{2}k\psi)+\frac{k}{r^3}[b^2-2a^2-2c^2+\tau c^2-(1+\lambda)(1+\zeta)(b^2+c^2)] &= 0\\ -3\lambda(3\mathfrak{A}_1{}^1-\tfrac{1}{2}k\psi)+\frac{k}{r^3}[a^2-2b^2+2c^2+\sigma c^2+(1+\lambda)(1+\zeta)(2b^2-c^2)] &= 0 \end{aligned}\right\} \quad . \quad (30).$$

Subtracting the second of these from the first and dividing by 9λ, we have

$$\mathfrak{A}_1{}^1-\mathbf{A}_1{}^1 = \frac{k}{3\lambda r^3}[a^2-b^2+b^2(1+\lambda)(1+\zeta)-\tfrac{1}{3}c^2(\tau-\sigma)] \quad . \quad . \quad . \quad . \quad (31).$$

Since we may write $3\mathsf{A}_1{}^1 - \frac{1}{2}k\psi$ in the form $-(\mathfrak{A}_1{}^1 - \mathsf{A}_1{}^1) - (\mathfrak{A}_1 - \mathsf{A}_1{}^1)$, the first of (30) in combination with (31) gives

$$\mathfrak{A}_1 - \mathsf{A}_1{}^1 = \frac{k}{3\lambda r^3}\left[a^2 + 2c^2 + c^2(1+\lambda)(1+\zeta) - \tfrac{1}{3}c^2(2\tau+\sigma)\right] \quad . \quad . \quad . \quad (32).$$

Referring to the values of τ and σ in (27), I find

$$-\tfrac{1}{3}(\tau-\sigma) = \tfrac{3}{14}\frac{k^2\kappa'^2}{r^2\kappa^2}\left[2 - 3(1+\kappa^2)\sin^2\gamma\right] + \tfrac{5}{56}\frac{k^4\kappa'^2}{r^4\kappa^4}\left[4(1+\kappa^2) - (5+6\kappa^2+5\kappa^4)\sin^2\gamma\right]$$

$$+ \tfrac{3}{10}\frac{K^2}{r^2\mathrm{K}^2}\left[2\mathrm{K}'^2\cos^2\gamma - \kappa'^2\sin^2\gamma(1+3\mathrm{K}^2)\right]$$

$$+ \tfrac{9}{56}\frac{K^4}{r^4\mathrm{K}^4}\left[4\mathrm{K}'^2(1+\mathrm{K}^2)\cos^2\gamma - \kappa'^2\sin^2\gamma(1+2\mathrm{K}^2+5\mathrm{K}^4)\right]$$

$$+ \tfrac{9}{28}\frac{k^2K^2}{r^4\kappa^2\mathrm{K}^2}\left[4(1-\kappa^2\mathrm{K}^2)\cos^2\gamma - \kappa'^2\sin^2\gamma(1+\kappa^2+\mathrm{K}^2+5\kappa^2\mathrm{K}^2)\right].$$

$$-\tfrac{1}{3}(2\tau+\sigma) = \tfrac{3}{14}\frac{k^2}{r^2\kappa^2}\left[7+5\kappa^2 - (3+\kappa^2)\sin^2\gamma\right] + \tfrac{5}{56}\frac{k^4}{r^4\kappa^4}\left[11+6\kappa^2+7\kappa^4 - (5+2\kappa^2+\kappa^4)\sin^2\gamma\right]$$

$$+ \tfrac{3}{10}\frac{K^2}{r^2\mathrm{K}^2}\left[7+5\mathrm{K}^2 - (3+\mathrm{K}^2)\sin^2\gamma\right]$$

$$+ \tfrac{9}{56}\frac{K^4}{r^4\mathrm{K}^4}\left[11+6\mathrm{K}^2+7\mathrm{K}^4 - (5+2\mathrm{K}^2+\mathrm{K}^4)\sin^2\gamma\right]$$

$$+ \tfrac{9}{28}\frac{k^2K^2}{r^4\kappa^2\mathrm{K}^2}\left[11+3\kappa^2+3\mathrm{K}^2+7\kappa^2\mathrm{K}^2 - (5+\kappa^2+\mathrm{K}^2+\kappa^2\mathrm{K}^2)\sin^2\gamma\right] \quad . \quad . \quad . \quad (33).$$

In all the cases which we shall have to consider the first of these expressions is small compared with the second, because κ is nearly equal to unity and κ' small, and because $\cos^2\gamma$ is also rather small.

Now let

$$\left.\begin{aligned} \epsilon &= -\tfrac{1}{3}(\tau-\sigma) + \zeta(1+\lambda)\cos^2\beta \\ \eta &= -\tfrac{1}{3}(2\tau+\sigma) + \zeta(1+\lambda) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33\ bis).$$

Then, since $a = c\cos\gamma$, $b = c\cos\beta$, the equations (31) and (32) become

$$\left.\begin{aligned} \mathfrak{A}_1{}^1 - \mathsf{A}_1{}^1 &= \frac{kc^2}{3\lambda r^3}(\cos^2\gamma + \lambda\cos^2\beta + \epsilon) \\ \mathfrak{A}_1 - \mathsf{A}_1{}^1 &= \frac{kc^2}{3\lambda r^3}(3+\lambda+\cos^2\gamma + \eta) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

Eliminating $kc^2/3\lambda r^3$, we have

$$(\mathfrak{A}_1 - \mathsf{A}_1{}^1)(\cos^2\gamma + \lambda\cos^2\beta + \epsilon) = (\mathfrak{A}_1{}^1 - \mathsf{A}_1{}^1)(3+\lambda+\cos^2\gamma+\eta) \quad . \quad . \quad (35).$$

This is the equation to be satisfied by the axes of the ellipsoid. If we treat ϵ and η as zero, it is the same as that found by ROCHE.*

* The form of this equation is so unlike ROCHE'S, that it may be worth while to prove the identity of the two.

ROCHE writes his equations in the form

$$\frac{st(t-s)}{(3+\lambda)t-\lambda s}\int_0^\infty \frac{u\,du}{(1+su)(1+tu)R} = \frac{s(1-s)}{s+3+\lambda}\int_0^\infty \frac{u\,du}{(1+su)(1+u)R} = \frac{t(1-t)}{t+\lambda}\int_0^\infty \frac{u\,du}{(1+u)(1+tu)R},$$

where $R^2 = (1+u)(1+su)(1+tu)$, and s is the square of the ratio of the least to the greatest axis, and t the square of the ratio of the least to the mean axis.

In my notation

$$s = \frac{a^2}{c^2} = \cos^2\gamma, \quad t = \frac{a^2}{b^2} = \frac{\cos^2\gamma}{\cos^2\beta}.$$

If we write $us+1 = \dfrac{\sin^2\gamma}{\sin^2\psi}$, and change the independent variable from u to ψ, we find

$$\left.\begin{aligned}
\text{ROCHE'S first integral} &= \frac{2\cos^3\beta}{\sin^5\gamma\cos^3\gamma}\int_0^\gamma \frac{\sin^2\psi(\sin^2\gamma-\sin^2\psi)}{\Delta^3}d\psi \\
\text{,, second ,,} &= \frac{2\cos\beta}{\sin^5\gamma\cos\gamma}\int_0^\gamma \frac{\sin^2\psi(\sin^2\gamma-\sin^2\psi)}{\cos^2\psi\Delta}d\psi \\
\text{,, third ,,} &= \frac{2\cos^3\beta}{\sin^5\gamma\cos\gamma}\int_0^\gamma \frac{\sin^2\psi(\sin^2\gamma-\sin^2\psi)}{\cos^2\psi\Delta^3}d\psi
\end{aligned}\right\} \quad \text{(A)},$$

where $\Delta^2 = 1-\kappa^2\sin^2\psi$.

The coefficients are

$$\left.\begin{aligned}
\frac{st(t-s)}{(3+\lambda)t-\lambda s} &= \frac{\cos^4\gamma\sin^2\beta}{\cos^2\beta[(3+\lambda)-\lambda\cos^2\beta]} \\
\frac{s(1-s)}{s+3+\lambda} &= \frac{\sin^2\gamma\cos^2\gamma}{3+\lambda+\cos^2\gamma} \\
\frac{t(1-t)}{t+\lambda} &= \frac{\kappa'^2\sin^2\gamma\cos^2\gamma}{\cos^2\beta(\cos^2\gamma+\lambda\cos^2\beta)}
\end{aligned}\right\} \quad \text{(B)}.$$

Then ROCHE'S equations are equivalent to

1st of (A) × 1st of (B) = 2nd of (A) × 2nd of (B) = 3rd of (A) × 3rd of (B).

But the two equations are not independent, and I will only pursue the consideration of the form involving the 2nd and 3rd of (A) and (B).

Now

$$\frac{\sin^2\psi(\sin^2\gamma-\sin^2\psi)}{\cos^2\psi\Delta} = \frac{\sin^2\psi}{\Delta} - \cos^2\gamma\frac{\tan^2\psi}{\Delta},$$

$$\frac{\sin^2\psi(\sin^2\gamma-\sin^2\psi)}{\cos^2\psi\Delta^3} = \frac{\cos^2\beta}{\kappa'^2}\frac{\sin^2\psi}{\Delta^3} - \frac{\cos^2\gamma}{\kappa'^2}\frac{\tan^2\psi}{\Delta},$$

and I have proved in (25) of the "Pear-shaped figure, &c." that

$$\mathfrak{A}_1 = \mathfrak{P}_1\mathfrak{Q}_1 = \frac{\kappa}{\sin^2\gamma}\int_0^\gamma \frac{\sin^2\psi}{\Delta}d\psi,$$

$$\mathsf{A}_1{}^1 = \mathsf{P}_1{}^1\mathsf{Q}_1{}^1 = \frac{\kappa\cos^2\gamma}{\sin^2\gamma}\int_0^\gamma \frac{\tan^2\psi}{\Delta}d\psi,$$

$$\mathfrak{A}_1{}^1 = \mathfrak{P}_1{}^1\mathfrak{Q}_1{}^1 = \frac{\kappa\cos^2\beta}{\sin^2\gamma}\int_0^\gamma \frac{\sin^2\psi}{\Delta^3}d\psi.$$

It is possible to express this equation in terms of elliptic integrals and to use LEGENDRE'S tables for finding the solution, but the method is very tedious, and after finding a few solutions in that way I abandoned it. It may, however, be worth while to mention that

$$\mathfrak{A}_1 - \mathbf{A}_1{}^1 = \frac{\kappa}{\sin^2\gamma}\left[-\frac{1}{\kappa'^2}\sin\gamma\cos\gamma\cos\beta + \frac{F}{\kappa^2} + \frac{\cos^2\beta - 2\kappa'^2}{\kappa^2\kappa'^2}E\right],$$

$$\mathfrak{A}_1{}^1 - \mathbf{A}_1{}^1 = \frac{\kappa}{\sin^2\gamma}\left[-\frac{2}{\kappa'^2}\sin\gamma\cos\gamma\cos\beta - \frac{\cos^2\beta}{\kappa^2}F + \frac{2\cos^2\beta - \kappa'^2}{\kappa^2\kappa'^2}E\right],$$

where

$$F = \int_0^\gamma \frac{d\psi}{\sqrt{(1-\kappa^2\sin^2\psi)}}, \quad E = \int_0^\gamma \sqrt{(1-\kappa^2\sin^2\psi)}\,d\psi.$$

When the forms of the ellipsoids have been determined, the radius vector becomes determinable from either of the equations (34).

The conditions that the internal potential of an ellipsoid satisfies POISSON'S equation and that ψ is homogeneous in a, b, c of degree -1, give the two following equations :—

$$\frac{\mathbf{A}_1{}^1}{a^2} + \frac{\mathfrak{A}_1{}^1}{b^2} + \frac{\mathfrak{A}_1}{c^2} - \frac{k}{abc} = 0,$$

$$\mathbf{A}_1{}^1 + \mathfrak{A}_1{}^1 + \mathfrak{A}_1 - \tfrac{1}{2}k\psi = 0.$$

Our two equations for $1/r^3$ may be written

$$\mathbf{A}_1{}^1 - \mathfrak{A}_1{}^1 \qquad + \frac{k}{3\lambda r^3}(a^2 + b^2\lambda + c^2\epsilon) = 0,$$

$$\mathbf{A}_1{}^1 \qquad - \mathfrak{A}_1 + \frac{k}{3\lambda r^3}[(3+\lambda)\,c^2 + a^2 + c^2\eta] = 0.$$

These four equations afford a determinant by which $\mathbf{A}_1{}^1$, $\mathfrak{A}_1{}^1$, $\mathfrak{A}_1$ may be eliminated. On reduction we find

$$\frac{1}{3\lambda r^3} = \frac{\tfrac{1}{2}\psi - \dfrac{3/abc}{1/a^2+1/b^2+1/c^2}}{3c^2 - a^2 + \lambda\,(b^2+c^2) + c^2\,(\eta+\epsilon) - \dfrac{6\,(1+\lambda+\tfrac{1}{2}\eta+\tfrac{1}{2}c^2\epsilon/b^2)}{1/a^2+1/b^2+1/c^2}}.$$

Therefore ROCHE'S second integral is equal to $\dfrac{2\cos\beta}{\kappa\sin^3\gamma\cos\gamma}(\mathfrak{A}_1 - \mathbf{A}_1{}^1)$, and his third integral is equal to $\dfrac{2\cos^3\beta}{\kappa\kappa'^2\sin^3\gamma\cos\gamma}(\mathfrak{A}_1{}^1 - \mathbf{A}_1{}^1)$.

Using these transformations of the second and third of (A), and dropping redundant factors, we get

$$(\cos^2\gamma + \lambda\cos^2\beta)\,(\mathfrak{A}_1 - \mathbf{A}_1{}^1) = (3 + \lambda + \cos^2\gamma)\,(\mathfrak{A}_1{}^1 - \mathbf{A}_1{}^1).$$

This agrees with the result in the text when ϵ and η are neglected.

On putting $a = c\cos\gamma$, $b = c\cos\beta$, and noting that $\psi = \frac{2F}{c\sin\gamma}$ and that $\frac{1+\lambda}{\lambda}c^3\cos\gamma\cos\beta = \mathrm{a}^3$, I find that

$$\frac{\mathrm{a}^3}{r^3} = \frac{\frac{3}{2}\left[2F\cot\gamma\cos\beta - \frac{6}{1+\sec^2\gamma+\sec^2\beta}\right]}{\frac{3-\cos^2\gamma+\lambda(1+\cos^2\beta)}{1+\lambda} - \frac{6}{1+\sec^2\gamma+\sec^2\beta} + \delta} \quad . \quad . \quad . \quad (36),*$$

where

$$\delta = \frac{1}{1+\lambda}\left[\eta+\epsilon - \frac{3(\eta+\epsilon\sec^2\beta)}{1+\sec^2\gamma+\sec^2\beta}\right].$$

However, this is not practically the most convenient form from which to compute the distance between the two ellipsoids.

* It is by no means obvious how this formula is consistent with results which we know by other means to be true. In the case when $\lambda = \infty$ we have a liquid planet rotating with the same angular velocity as an infinitely small satellite revolving in a circular orbit in its equator.

Let us first consider the value of ζ. In the present case the semi-axes A, B, C pertain to the infinitely small satellite, and are therefore negligible compared with terms in a, b, c. Since the axis denoted by c is that coincident with the satellite's radius vector, and since the equatorial plane of the planet must have a circular section, we have $c = b$.

But since $b = c\cos\beta$, it follows that $\beta = 0$ or $\kappa\sin\gamma = 0$. Now γ does not vanish for $a = c\cos\gamma$, and a is the polar semi-radius of the planet; therefore $\kappa = 0$.

If we consider the formula (24) for ζ, expressing, however, the several terms in the form of (15), we see that for $\lambda = \infty$

$$\zeta = \frac{3}{10r^2}(c^2-a^2) + \frac{3}{56r^4}\cdot 3(c^2-a^2)^2 + \frac{1}{48r^6}\cdot 5(c^2-a^2)^3\ldots,$$

whence

$$\zeta = \frac{3c^2}{10r^2}\sin^2\gamma + \frac{9c^4}{56r^4}\sin^4\gamma + \frac{5c^6}{48r^6}\sin^6\gamma\ldots \quad . \quad . \quad . \quad (a).$$

The factor of correction to KEPLER'S law of periodic times for a small satellite revolving about an oblate planet, whose equatorial radius is c and whose eccentricity of figure is $\sin\gamma$, is $1+\zeta$, where ζ is expressed by the above series (a).

Now considering the formula (33 *bis*), we see that for $\lambda = \infty$ and $\beta = 0$

$$\frac{\epsilon}{1+\lambda} = \zeta\cos^2\beta = \zeta; \quad \frac{\eta}{1+\lambda} = \zeta.$$

In (36) we therefore have

$$\delta = \frac{1}{1+\lambda}\left[\eta+\epsilon - \frac{3(\eta+\epsilon\sec^2\beta)}{1+\sec^2\beta+\sec^2\gamma}\right] = 2\zeta\left[1 - \frac{3}{3+\tan^2\gamma}\right] = \frac{2\zeta\tan^2\gamma}{3+\tan^2\gamma}.$$

When $\kappa = 0$, the elliptic integral F is equal to γ; thus (36) becomes

$$\frac{\mathrm{a}^3}{r^3} = \frac{\frac{3}{2}\left[2\gamma\cot\gamma - \frac{6}{3+\tan^2\gamma}\right]}{2 - \frac{6}{3+\tan^2\gamma} + \frac{2\zeta\tan^2\gamma}{3+\tan^2\gamma}}.$$

§ 11. *Solution of the Equations.*

In all the ellipsoids of which we shall have to find the axes, it happens that $\kappa'^2 \tan^2 \gamma$ is fairly small compared with unity. Hence it is possible to expand Δ in powers of that quantity.

We have

$$\Delta^2 = 1 - \kappa^2 \sin^2 \gamma = \cos^2 \gamma \, (1 + \kappa'^2 \tan^2 \gamma),$$

and

$$\frac{1}{\Delta} = \frac{1}{\cos \gamma} \left[1 - \tfrac{1}{2} \kappa'^2 \tan^2 \gamma + \tfrac{1 \cdot 3}{2 \cdot 4} \kappa'^4 \tan^4 \gamma - \ldots\right],$$

$$\frac{1}{\Delta^3} = \frac{1}{\cos^3 \gamma} \left[1 - \tfrac{3}{2} \kappa'^2 \tan^2 \gamma + \tfrac{3 \cdot 5}{2 \cdot 4} \kappa'^4 \tan^4 \gamma - \ldots\right].$$

From this we easily obtain

$$\frac{\omega^2}{2\pi\rho} = \tfrac{2}{3}(1+\zeta)\frac{\mathrm{a}^3}{r^3} = \cot^3 \gamma \left[\gamma\,(3 + \tan^2 \gamma) - 3 \tan \gamma\right].$$

This is the well-known formula for the angular velocity of MACLAURIN'S ellipsoid.

It should be remarked that (35) is identically satisfied by $\lambda = \infty$, $\kappa = 0$, for when we use the above values of ϵ and η, the equation becomes divisible by $1 + \zeta$.

Since ζ is a symmetrical function of a, b, c and A, B, C, it follows that ζ is the same in form for λ and for $1/\lambda$. Therefore when we consider the case of $\lambda = 0$, the formula (a) gives the required result, but c and γ refer to the large body which is throughout most of this paper indicated by capital letters.

Thus for $\lambda = 0$,

$$\zeta = \tfrac{3}{10}\frac{C^2}{r^2}\sin^2\Gamma + \tfrac{9}{56}\left(\frac{C^2}{r^2}\sin^2\Gamma\right)^2 + \tfrac{5}{48}\left(\frac{C^2}{r^2}\sin^2\Gamma\right)^3 \ldots$$

$$= \tfrac{3}{10}\frac{\mathrm{a}^2}{r^2}\frac{\sin^2\Gamma}{\cos^{2/3}\Gamma} + \tfrac{9}{56}\frac{\mathrm{a}^4}{r^4}\left(\frac{\sin^2\Gamma}{\cos^{2/3}\Gamma}\right)^2 + \tfrac{5}{48}\frac{\mathrm{a}^6}{r^6}\left(\frac{\sin^2\Gamma}{\cos^{2/3}\Gamma}\right)^3 \ldots$$

In the case of $\lambda = 0$, k vanishes; K also vanishes and so also does the angle B. Hence we have

$$\epsilon = \zeta \cos^2 \beta, \quad \eta = \zeta.$$

With these values, equation (35) becomes

$$(\mathfrak{A}_1 - \mathsf{A}_1{}^1)\left[\cos^2\gamma + (\lambda+\zeta)\cos^2\beta\right] = (\mathfrak{A}_1{}^1 - \mathsf{A}_1{}^1)\left[3 + (\lambda+\zeta) + \cos^2\gamma\right].$$

Hence ζ plays the part of an augmentation to λ.

With $\lambda = 0$ the equation assumes the form

$$(\mathfrak{A}_1 - \mathsf{A}_1{}^1)(\cos^2\gamma + \zeta\cos^2\beta) = (\mathfrak{A}_1{}^1 - \mathsf{A}_1{}^1)(3 + \zeta + \cos^2\gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b).$$

It follows therefore that an infinitesimal satellite revolving about an oblate planet, whose rotation is the same as the revolution of the satellite, is very nearly identical in form with a small but finite satellite whose mass is a fraction of a *spherical* planet expressed by ζ. This curious conclusion follows from the fact that if we take equation (35) and put ϵ and η zero (which corresponds to a spherical planet and small satellite), we get exactly the equation (b) just found, only with λ in place of ζ.

Now from (25) of the "Pear-shaped figure,"

$$\mathfrak{A}_1 = \frac{\kappa}{\sin^2\gamma}\int_0^\gamma \frac{\sin^2\gamma}{\Delta}\,d\gamma,\quad \mathsf{A}_1{}^1 = \frac{\kappa\cos^2\gamma}{\sin^2\gamma}\int_0^\gamma \frac{\tan^2\gamma}{\Delta}\,d\gamma,\quad \mathfrak{A}_1{}^1 = \frac{\kappa\cos^2\beta}{\sin^2\gamma}\int_0^\gamma \frac{\sin^2\gamma}{\Delta^3}\,d\gamma.$$

When the Δ's under the integral signs are expanded, all the terms of the series involve integrals of one of two types. If we write

$$\Omega = \frac{1}{\sin\gamma}\log_e\frac{1+\sin\gamma}{\cos\gamma},$$

the types are

$$\int_0^\gamma \frac{\sin^{2n}\gamma}{\cos^{2n-1}\gamma}d\gamma = \sin\gamma\Big[\frac{1}{2n-2}\tan^{2n-2}\gamma - \frac{2n-1}{(2n-2)(2n-4)}\tan^{2n-4}\gamma + \ldots$$
$$+(-)^n\frac{(2n-1)(2n-3)\ldots5}{(2n-2)(2n-4)\ldots2}\tan^2\gamma + (-)^{n+1}\frac{(2n-1)\ldots3}{(2n-2)\ldots2}(\Omega-1)\Big],$$

$$\int_0^\gamma \frac{\sin^{2n}\gamma}{\cos^{2n+1}\gamma}d\gamma = \sin\gamma\Big[\frac{1}{2n}\tan^{2n}\gamma - \frac{1}{2n(2n-2)}\tan^{2n-2}\gamma + \frac{2n-1}{2n(2n-2)(2n-4)}\tan^{2n-4}\gamma - \ldots$$
$$-(-)^n\frac{(2n-1)(2n-3)\ldots5}{2n(2n-2)\ldots2}\tan^2\gamma - (-)^{n+1}\frac{(2n-1)\ldots3}{2n(2n-2)\ldots2}(\Omega-1)\Big].$$

As it is not quite obvious what interpretation is to be put on these formulæ for the smaller values of n, I may mention that when $n = 0, 1, 2$ respectively, the first integral is $\sin\gamma$; $\sin\gamma(\Omega-1)$; $\sin\gamma[\frac{1}{2}\tan^2\gamma - \frac{3}{2}(\Omega-1)]$, and the second is $\sin\gamma\,\Omega$; $\sin\gamma[\frac{1}{2}\tan^2\gamma - \frac{1}{2}(\Omega-1)]$; $\sin\gamma[\frac{1}{4}\tan^4\gamma - \frac{1}{4.2}\tan^2\gamma + \frac{3}{4.2}(\Omega-1)]$. For larger values of n the interpretation is obvious.

If we use these integrals and write

$$\left.\begin{aligned}\frac{\sin\gamma}{\kappa}(\mathfrak{A}_1{}^1 - \mathsf{A}_1{}^1) &= \kappa'^2[\sigma_0 - \sigma_1\kappa'^2 + \sigma_2\kappa'^4 - \sigma_3\kappa'^6\ldots]\\ \frac{\sin\gamma}{\kappa}(\mathfrak{A}_1 - \mathsf{A}_1{}^1) &= \tau_0 - \tau_1\kappa'^2 + \tau_2\kappa'^4 - \tau_3\kappa'^6\ldots\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (37),$$

we find

$$\sigma_0 = \tfrac{1}{8}[2\tan^2\gamma + 3 - (3+\sin^2\gamma)\Omega]$$
$$\sigma_1 = \tfrac{3}{32}[\tfrac{4}{3}\tan^4\gamma - \tfrac{8}{3}\tan^2\gamma - 5 + (5+\sin^2\gamma)\Omega]$$
$$\sigma_2 = \tfrac{75}{1024}[\tfrac{16}{15}\tan^6\gamma - \tfrac{8}{5}\tan^4\gamma + \tfrac{10}{3}\tan^2\gamma + 7 - (7+\sin^2\gamma)\Omega]$$
$$\sigma_3 = \tfrac{245}{4096}[\tfrac{32}{35}\tan^8\gamma - \tfrac{128}{105}\tan^6\gamma + \tfrac{28}{15}\tan^4\gamma - 4\tan^2\gamma - 9 + (9+\sin^2\gamma)\Omega]$$
$$\tau_0 = \tfrac{1}{2}[-3 + (3-\sin^2\gamma)\Omega]$$
$$\tau_1 = \tfrac{3}{16}[\tfrac{2}{3}\tan^2\gamma + 5 - (5-\sin^2\gamma)\Omega]$$
$$\tau_2 = \tfrac{15}{128}[\tfrac{4}{15}\tan^4\gamma - \tfrac{4}{3}\tan^2\gamma - 7 + (7-\sin^2\gamma)\Omega]$$
$$\tau_3 = \tfrac{175}{2048}[\tfrac{16}{105}\tan^6\gamma - \tfrac{8}{15}\tan^4\gamma + 2\tan^2\gamma + 9 - (9-\sin^2\gamma)\Omega]$$
$$\tau_4 = \tfrac{5.3^2.7^2}{2^{15}}[\tfrac{32}{315}\tan^8\gamma - \tfrac{32}{105}\tan^6\gamma + \tfrac{4}{5}\tan^4\gamma - \tfrac{4}{3}\tan^2\gamma - 11 + (11-\sin^2\gamma)\Omega]. \quad (38).$$

It would not be difficult to find the general expressions for these functions, but it does not seem worth while to do so.

The equation (35) for determining the form of the ellipsoid involves the factor $\cos^2\gamma + \lambda\cos^2\beta + \epsilon$; if we write

$$M = \frac{\lambda \sin^2\gamma}{(1+\lambda)\cos^2\gamma + \epsilon},$$

this factor may be written in the form $[(1+\lambda)\cos^2\gamma + \epsilon][1 + M\kappa'^2]$. Hence the equation (35) may be written

$$[\tau_0 - \tau_1\kappa'^2 + \tau_2\kappa'^4 - \tau_3\kappa'^6 \ldots][(1+\lambda)\cos^2\gamma + \epsilon][1 + M\kappa'^2]$$
$$= \kappa'^2[\sigma_0 - \sigma_1\kappa'^2 + \sigma_2\kappa'^4 - \sigma_3\kappa'^6 \ldots][3 + \lambda + \cos^2\gamma + \eta].$$

If now we put

$$v_0 = \frac{\tau_0}{\sigma_0}, \quad v_1 = \frac{\sigma_1}{\sigma_0} - \frac{\tau_1}{\tau_0}, \quad v_2 = \frac{\sigma_2}{\sigma_0} - \frac{\tau_2}{\tau_0} - \frac{\sigma_1}{\sigma_0}v_1, \quad v_3 = \frac{\sigma_3}{\sigma_0} - \frac{\tau_3}{\tau_0} - \frac{\sigma_1}{\sigma_0}v_2 - \frac{\sigma_2}{\sigma_0}v_1,$$

we have

$$\frac{\tau_0 - \tau_1\kappa'^2 + \tau_2\kappa'^4 \ldots}{\sigma_0 - \sigma_1\kappa'^2 + \sigma_2\kappa'^4 \ldots} = v_0[1 + v_1\kappa'^2 - v_2\kappa'^4 + v_3\kappa'^6 - \ldots].$$

Hence our equation may be written

$$\frac{v_0[(1+\lambda)\cos^2\gamma + \epsilon]}{3 + \lambda + \cos^2\gamma + \eta}(1 + v_1\kappa'^2 - v_2\kappa'^4 + v_3\kappa'^6 \ldots)(1 + M\kappa'^2) = \kappa'^2.$$

Whence on writing

$$L = \frac{3 + \lambda + \cos^2\gamma + \eta}{(1+\lambda)\cos^2\gamma + \epsilon}$$

$$\kappa'^2 = \frac{1 + (Mv_1 - v_2)\kappa'^4 - (Mv_2 - v_3)\kappa'^6 \ldots}{L/v_0 - M - v_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39).$$

The determination of L for given value of γ involves that of η and ϵ, and these can only be found from an approximate preliminary solution of the whole problem. But when L is known approximately, the solution of (39) is very simple, for we first neglect the terms in κ'^4 and κ'^6 on the right-hand side, and so determine a first approximation to κ'. As a fact I have not included the term in κ'^6 in my computations, because it would not make so much as 1′ difference in the value of $\cos^{-1}\kappa'$.

For ROCHE's problem when ϵ and η are neglected the solution is very short, but when these terms are included the computation is laborious.

We now turn to the determination of the radius vector.

We have

$$\mathfrak{A}_1 - \mathbf{A}^1_1 = \frac{kc^2}{3\lambda r^3}(3 + \lambda + \cos^2\gamma + \eta).$$

Since $c^2 = \dfrac{k^2}{\sin^2\beta}$ and $\dfrac{k^3\cos\gamma\cos\beta}{\sin^3\beta} = \dfrac{\lambda a^3}{1+\lambda}$, we have

$$\frac{kc^2}{3\lambda r^3} = \frac{1}{3(1+\lambda)}\,\frac{\kappa\sin\gamma}{\cos\beta\cos\gamma}\cdot\frac{a^3}{r^3}.$$

Therefore

$$\frac{\kappa}{\sin\gamma}(\tau_0-\tau_1\kappa'^2+\ldots) = \frac{\kappa\sin\gamma}{3(1+\lambda)\cos\beta\cos\gamma}\frac{a^3}{r^3}(3+\lambda+\cos^2\gamma+\eta).$$

Whence

$$\frac{a^3}{r^3} = 3(1+\lambda)\frac{\cos\beta\cos\gamma}{\sin^2\gamma}\frac{\tau_0-\tau_1\kappa'^2+\tau_2\kappa'^4-\tau_3\kappa'^6\ldots}{3+\lambda+\cos^2\gamma+\eta} \quad . \quad . \quad . \quad . \quad . \quad (40).$$

Thus a table of values of τ_0, τ_1, τ_2, τ_3 enables us to compute r for a given value of γ, when κ' has been found.

In the following tables the v's and τ's were computed for the even degrees of γ and interpolated for the odd degrees. These functions are found as the differences between large numbers, and therefore great care would be required to determine them

TABLE of Auxiliary Functions.

γ.	Log v_0.	v_1.	Log v_1.	Log v_2.	Log v_3.	τ_0.	Log τ_1.	Log τ_2.	Log τ_3.
30°	9·94493	0·14279	9·15470	7·9576	7·15	0·010592	6·85440	5·9805	5·222
31	9·94085	0·15461	9·18925	8·0264	7·25	0·012124	6·94587	6·1063	5·378
32	9·93661	0·16722	9·22329	8·0941	7·36	0·013824	7·03507	6·2288	5·533
33	9·93222	0·18065	9·25684	8·1607	7·47	0·015702	7·12194	6·3478	5·685
34	9·92764	0·19493	9·28988	8·2263	7·58	0·017774	7·20666	6·4639	5·834
35	9·92285	0·21006	9·32234	8·2906	7·67	0·020052	7·28937	6·5780	5·979
36	9·91786	0·22613	9·35436	8·3541	7·76	0·022554	7·37022	6·6899	6·122
37	9·91269	0·24327	9·38608	8·4173	7·87	0·025294	7·44939	6·7994	6·262
38	9·90731	0·26150	9·41747	8·4800	7·97	0·028290	7·52697	6·9069	6·400
39	9·90167	0·28091	9·44857	8·5418	8·06	0·031559	7·60306	7·0127	6·536
40	9·89578	0·30160	9·47943	8·6031	8·14	0·035121	7·67777	7·1167	6·669
41	9·88965	0·32368	9·51011	8·6646	8·23	0·038994	7·75120	7·2192	6·802
42	9·88327	0·34724	9·54063	8·7259	8·32	0·043203	7·82342	7·3202	6·932
43	9·87665	0·37244	9·57105	8·7869	8·42	0·047768	7·89452	7·4200	7·061
44	9·86974	0·39937	9·60138	8·8476	8·51	0·052713	7·96459	7·5186	7·189
45	9·86257	0·42823	9·63168	8·9081	8·60	0·058064	8·03371	7·6162	7·316
46	9·85509	0·45917	9·66197	8·9685	8·69	0·063847	8·10195	7·7128	7·442
47	9·84729	0·49239	9·69231	9·0290	8·79	0·070093	8·16939	7·8087	7·567
48	9·83915	0·52812	9·72274	9·0896	8·87	0·076830	8·23608	7·9036	7·691
49	9·83069	0·56656	9·75325	9·1505	8·96	0·084093	8·30210	7·9981	7·814
50	9·82187	0·60804	9·78393	9·2117	9·05	0·091916	8·36752	8·0920	7·938
51	9·81263	0·65278	9·81477	9·2733	9·15	0·100336	8·43241	8·1855	8·061
52	9·80300	0·70121	9·84585	9·3353	9·24	0·109394	8·49682	8·2786	8·183
53	9·79298	0·75372	9·87721	9·3976	9·33	0·119134	8·56082	8·3716	8·306
54	9·78252	0·81075	9·90889	9·4606	9·43	0·129601	8·62447	8·4643	8·429
55	9·77159	0·87288	9·94095	9·5245	9·52	0·140845	8·68785	8·5570	8·552
56	9·76016	0·94064	9·97342	9·5893	9·62	0·152919	8·75103	8·6500	8·676
57	9·74819	1·01471	·00634	9·6550	9·72	0·165883	8·81407	8·7431	8·801
58	9·73565	1·09590	·03977	9·7216	9·82	0·179801	8·87703	8·8367	8·926
59	9·72252	1·18517	·07378	9·7893	9·92	0·194740	8·93999	8·9306	9·052
60	9·70874	1·28362	·10844	9·8582	·02	0·210779	9·00303	9·0252	9·179
61	9·69426	1·39268	·14385	9·9287	·13	0·227997	9·06626	9·1206	9·308
62	9·67910	1·51349	·17998	·0006	·24	0·246485	9·12966	9·2168	9·439
63	9·66319	1·6480	·21696	·0743	·36	0·266343	9·19331	9·3140	9·572

with a very high degree of accuracy. The differences of the tabulated numbers do not run with perfect smoothness, showing that there are residual errors of one or two units in the last place of decimals. The accuracy is however amply sufficient for the end in view, and it would have been wasteful to spend more time over the computations.

§ 12. *Determination of the Form of the Second Ellipsoid.*

The parameters Γ and K determine the form of the second ellipsoid in the same way that γ and κ determine the first. It is obvious that a/r is determinable in two ways, and therefore any given value of γ must correspond to a certain definite value of Γ. The fitting together of the two solutions can only be effected with accuracy by interpolation, but it would be so enormously laborious to find by mere conjecture the region in which to begin calculating with assumed values of Γ, that an approximate solution of the problem becomes a practical necessity.

After various trials I find that on neglecting ϵ and η and writing

$$\chi = \frac{\sin^2\gamma}{7(3+\lambda+\cos^2\gamma)}$$

the solution for κ' may be written approximately in the form

$$\kappa'^2 = \frac{(1+\lambda)\cos^2\gamma}{3+\lambda+\cos^2\gamma}\left[1-(9-7\lambda)\chi-(111+228\lambda-49\lambda^2)\chi^2 - \tfrac{1}{11}(7875+28185\lambda+42333\lambda^2-3773\lambda^3)\chi^3\ldots\right] \quad . \quad . \quad (41).$$

If ϵ were added to the numerator of the factor outside the bracket, and η to the denominator, this formula would give nearly as good results as the more accurate method of the last section.

Also I find

$$\frac{a^3}{r^3} = \frac{\frac{2}{5}(1+\lambda)\cos\beta\cos\gamma\sin^2\gamma}{3+\lambda+\cos^2\gamma}\left[1+\tfrac{9}{2}(5+\lambda)\chi+\tfrac{1}{8}(3055+1718\lambda+175\lambda^2)\chi^2 + \frac{1}{2^4.3.11}(3533389+3222607\lambda+1021479\lambda^2+60025\lambda^3)\chi^3+\ldots\right] \quad . \quad . \quad (42).$$

In order to obtain the desired approximation, it is necessary to express a^3/r^3 by a series which can be inverted; but this is not possible in the form just given, because $\cos\beta$ depends on κ and therefore involves χ. I find then by means of the above series for κ'^2 that

$$\cos\beta\cos\gamma = 1-\tfrac{7}{2}(7+\lambda)\chi+\tfrac{7}{8}(69-106\lambda-7\lambda^2)\chi^2+\tfrac{7}{16}(253+753\lambda-1901\lambda^2-49\lambda^3)\chi^3\ldots$$

On introducing this in the above formula for a^3/r^3, I find

$$\frac{a^3}{r^3} = \tfrac{14}{5}(1+\lambda)\chi[1-(2-\lambda)\chi-(109+67\lambda+0\lambda^2)\chi^2 - \frac{1}{2^2.3.11}(157712+261395\lambda+97656\lambda^2+1568\lambda^3)\chi^3\ldots].$$

On writing

$$\alpha = \tfrac{5}{14}\cdot\frac{1}{1+\lambda}\frac{a^3}{r^3},$$

and inverting the series we find

$$\chi = \alpha+(2-\lambda)\alpha^2+(117+59\lambda+2\lambda^2)\alpha^3 + \frac{1}{2^2.3.11}(306872+269975\lambda+5739\lambda^2+908\lambda^3)\alpha^4\ldots \quad . \quad . \quad . \quad (43).$$

This series expresses a function of γ in series proceeding by powers of a^3/r^3, and a similar series must also connect a function of Γ with the radius vector, so as to determine the figure of the second ellipsoid appropriately. This second series may be written down by symmetry.

Since λ must now be replaced by $1/\lambda$, the function corresponding to α is $\tfrac{5}{14}\dfrac{\lambda}{1+\lambda}\dfrac{a^3}{r^3}$ or $\lambda\alpha$, and the function corresponding to χ is $\dfrac{\lambda\sin^2\Gamma}{7[(3+\cos^2\Gamma)\lambda+1]}$.

If then we write

$$X = \frac{\sin^2\Gamma}{7[(3+\cos^2\Gamma)\lambda+1]},$$

the symmetrical series for the other ellipsoid is

$$X = \alpha+(2\lambda-1)\alpha^2+(117\lambda^2+59\lambda+2\lambda^2)\alpha^3 + \frac{1}{2^2.3.11}(306872\lambda^3+269975\lambda^2+5739\lambda^3+908)\alpha^4\ldots$$

Now α is easily computed for the first ellipsoid, and then X is computed by the series. Thus we have

$$\sin^2\Gamma = \frac{(4\lambda+1)X}{\lambda X+\frac{1}{7}}.$$

We obtain in this way a fairly accurate value of Γ corresponding to the value of γ which determines the first ellipsoid. We can then compute K' by the method of the last section. We may thus obtain a good idea of the values of Γ and K with which it is necessary to work in order to obtain the final solution.

§ 13. *The Equilibrium of Two Ellipsoids joined by a Weightless Pipe.*

In § 3 the problem is considered of the equilibrium of two masses of liquid, each constrainedly spherical, when joined by a pipe without weight. It was shown that the condition determining the ratio of the masses for given radius vector is expressed by a certain equation which was written $f(\mathrm{a}/r) = 0$. Further, it was proved that if $f(\mathrm{a}/r)$ is positive the two spheres of liquid are too far apart to admit of junction, and if it is negative they are too near. Finally we found that all these solutions were unstable.

The solutions for the two spheres showed them to be always very close together, and as all the solutions for two ellipsoids, when they are in limiting stability, made them much further apart than were the two spheres, it seemed somewhat improbable that two ellipsoids could be similarly joined by a pipe, and certain that they would be unstable if such junction were possible. Nevertheless, it seemed conceivable that the additional terms, which must appear in $f(\mathrm{a}/r)$ when the constraint to spherical form is removed, might alter the conditions so that the junction of ellipsoids by a pipe should become possible. It thus became expedient to solve a problem analogous to that of § 3 when the two masses of liquid are ellipsoidal.

The conditions of equilibrium of two ellipsoids unjoined by a pipe are given in § 10, and the additional condition corresponding to junction by a pipe is

$$\frac{dV}{d\lambda} + \tfrac{1}{2}\omega^2 \frac{dI}{d\lambda} = 0.$$

In the present investigation I shall neglect the higher ellipticities, denoted f_i^s and F_i^s, and terms of higher order than those in $1/r^5$.

With this degree of approximation we have

$$V = (\tfrac{4}{3}\pi\rho \mathrm{a}^3)^2 \left[\frac{\lambda}{(1+\lambda)^2 r} + \tfrac{3}{10}\frac{\lambda^2}{(1+\lambda)^2}\psi + \tfrac{3}{10}\frac{1}{(1+\lambda)^2}\Psi\right] + (eE)_2,$$

where $(eE)_2$ is given in (14) (with omission of terms in $1/r^7$),

Also

$$\psi = \frac{2}{k}\int_{\nu_0}^{\infty} \frac{d\nu}{(\nu^2-1)^{1/2}(\nu^2-1/\kappa^2)^{1/2}},$$

and Ψ has a symmetrical form in K and K.

We have besides $\omega^2 = \frac{4}{3}\pi\rho\frac{\mathrm{a}^3}{r^3}(1+\zeta)$, where ζ is given in (24); and I is given in (22).

The differentiation with respect to λ and the subsequent re-arrangement of the equation are rather tedious, and I will not give the details of the operations. It may however be well to note that k, K are functions of λ, and that

$$\frac{d\psi}{d\lambda} = -\frac{1}{3\lambda(1+\lambda)}\psi, \quad \frac{d\Psi}{d\lambda} = \frac{1}{3(1+\lambda)}\Psi.$$

I find finally that the equation of condition is $f\left(\frac{a}{r}\right) = 0$, where

$$f\left(\frac{a}{r}\right) = -\tfrac{9}{2}\frac{1+\lambda^{1/3}+\lambda^{2/3}}{(1+\lambda)^{1/3}(1+\lambda^{1/3})}\frac{a}{r} - \frac{3a(\lambda\psi-\Psi)}{2(1+\lambda)^{1/3}(1-\lambda^{2/3})}$$
$$-\frac{1}{(1+\lambda)^{1/3}(1-\lambda^{2/3})}\Big[\frac{ac^2}{20r^3}\{48-20\lambda-(19-15\lambda)\cos^2\gamma-(9-25\lambda)\cos^2\beta\}$$
$$+\frac{aC^2}{20r^3}\{20-48\lambda-(15-19\lambda)\cos^2\Gamma-(25-9\lambda)\cos^2 B\}$$
$$+\frac{3ac^4}{560r^5}\{288-112\lambda-(260-140\lambda)\cos^2\gamma-(204-196\lambda)\cos^2\beta$$
$$+(87-63\lambda)\cos^4\gamma+(59-91\lambda)\cos^4\beta$$
$$+(30-70\lambda)\cos^2\beta\cos^2\gamma\}$$
$$+\frac{3aC^4}{560r^5}\{112-288\lambda-(140-260\lambda)\cos^2\Gamma-(196-204\lambda)\cos^2 B$$
$$+(63-87\lambda)\cos^4\Gamma+(91-59\lambda)\cos^4 B$$
$$+(70-30\lambda)\cos^2 B\cos^2\Gamma\}$$
$$+\frac{3ac^2C^2}{40r^5}\{40(1-\lambda)-(18-22\lambda)\cos^2\gamma-(22-18\lambda)\cos^2\Gamma$$
$$-(14-26\lambda)\cos^2\beta-(26-14\lambda)\cos^2 B$$
$$+15(1-\lambda)(\cos^2\gamma\cos^2\Gamma+\cos^2\beta\cos^2 B)$$
$$+(7-3\lambda)\cos^2\gamma\cos^2 B+(3-7\lambda)\cos^2\Gamma\cos^2\beta\}\Big] \quad (44).$$

In this expression c and C are respectively the longest semi-axes of the two ellipsoids, which are pointed at one another.

We may derive ψ and Ψ from LEGENDRE's tables of elliptic integrals for

$$\psi = \frac{2}{c\sin\gamma}F(\kappa, \gamma), \quad \Psi = \frac{2}{2C\sin\Gamma}F(K, \Gamma);$$

or we may expand the integrals in powers of $\kappa'^2\tan^2\gamma$ and obtain the approximate formula

$$\psi = \frac{2}{c}[1+(\Omega-1)(1+\tfrac{1}{4}\kappa'^2+\tfrac{9}{64}\kappa'^4+\tfrac{25}{256}\kappa'^6\ldots)-\tfrac{1}{4}\kappa'^2\tan^2\gamma(1+\tfrac{3}{16}\kappa'^2+\tfrac{25}{192}\kappa'^4\ldots)$$
$$+\tfrac{3}{32}\kappa'^4\tan^4\gamma(1+\tfrac{5}{36}\kappa'^2\ldots)-\tfrac{5}{96}\kappa'^6\tan^6\gamma(1+\ldots)] \quad . \quad (45),$$

where $\Omega = \frac{1}{\sin\gamma}\log_e\frac{1+\sin\gamma}{\cos\gamma}$. The formula for Ψ is of course symmetrical.

It should be noted that when the two ellipsoids reduce to spheres, we have

$$\gamma=\beta=\Gamma=B=0,\; c=\frac{\lambda^{1/3}a}{(1+\lambda)^{1/3}},\; C=\frac{1}{(1+\lambda)^{1/3}}a,\; \psi=\frac{2}{c}=\frac{2}{a}\frac{(1+\lambda)^{1/3}}{\lambda^{1/3}},\; \Psi=\frac{2}{a}(1+\lambda)^{1/3}.$$

Thus

$$f\left(\frac{\mathrm{a}}{r}\right) = -\tfrac{9}{2}\frac{1+\lambda^{1/3}+\lambda^{2/3}}{(1+\lambda)^{1/3}(1+\lambda^{1/3})}\frac{\mathrm{a}}{r}+3+\frac{\mathrm{a}^3}{r^3}.$$

This is the form obtained in the solution of the restricted problem of § 3.

We conclude that if $f(\mathrm{a}/r)$, as expressed in (44), with values derived from any solution of the problem of the equilibrium of two ellipsoids unconnected by a pipe, is positive, the two figures are too far apart to admit of junction, and *vice versâ*. I have in fact always found it positive, although always diminishing as r diminishes, so that junction would seem to be always impossible, at least so long as the approximation retains any validity. This might indicate that there is no figure of equilibrium shaped like an hour-glass with a thin neck. However, I return to this subject in discussing numerical solutions, and in the summary of results.

In the case where the two masses are equal, $\lambda=1$, and the above formula for $f(\mathrm{a}/r)$ fails by becoming indeterminate. As this is a case of especial interest, it must be considered.

Since the two shapes are now exactly alike, we may take κ, γ, β, k to define either of them.

When $\lambda = 1$, the first term of $f(\mathrm{a}/r)$ becomes $-\dfrac{27}{4.2^{1/3}}\dfrac{\mathrm{a}}{r}$.

The second term becomes

$$-\frac{3\mathrm{a}\psi(\lambda-1)}{2(1+\lambda)^{1/3}(1-\lambda^{2/3})} = \frac{3\psi\mathrm{a}(1+\lambda^{1/3}+\lambda^{2/3})}{2(1+\lambda)^{1/3}(1+\lambda^{1/3})} = \frac{9\psi\mathrm{a}}{4.2^{1/3}}.$$

All the terms in $1/r^3$ are one of the two forms

$$F\left[\frac{c^2-\lambda C'^2}{1-\lambda^{2/3}}\right] \quad \text{or} \quad G\left[\frac{\lambda c^2-C'^2}{1-\lambda^{2/3}}\right].$$

Now,

$$c^2 = \frac{\mathrm{a}^2}{(\cos\beta\cos\gamma)^{2/3}}\frac{\lambda^{2/3}}{(1+\lambda)^{2/3}}, \quad C'^2 = \frac{\mathrm{a}^2}{(\cos \mathrm{B}\cos \Gamma)^{2/3}}\frac{1}{(1+\lambda)^{2/3}}.$$

In the limit $\beta = \mathrm{B}$, $\gamma = \Gamma$, and $\lambda = 1$; and we find that the first of these forms becomes $\tfrac{1}{2}Fc^2$, and the second $-\tfrac{5}{2}Gc^2$.

Again, of the terms in $1/r^5$, those in c^4 and C'^4 are of one of the two forms

$$F\left[\frac{c^4-\lambda C'^4}{1-\lambda^{2/3}}\right] \quad \text{or} \quad G\left[\frac{\lambda c^4-C'^4}{1-\lambda^{2/3}}\right].$$

In the limit the first of these reduces to $-\tfrac{1}{2}Fc^4$, and the second to $-\tfrac{7}{2}Gc^4$.

The last term in $1/r^7$ has a common factor $1-\lambda$, when $\gamma = \Gamma$, $\beta = \mathrm{B}$, and

$$\frac{1-\lambda}{(1+\lambda)^{1/3}(1-\lambda^{2/3})} = \frac{1+\lambda^{1/3}+\lambda^{2/3}}{(1+\lambda)^{1/3}(1+\lambda^{1/3})} = \frac{3}{2.2^{1/3}}.$$

By means of these transformations we find for $\lambda = 1$

$$f\left(\frac{\mathrm{a}}{r}\right) = \frac{3}{16^{1/3}}\left[-\tfrac{9}{2}\frac{\mathrm{a}}{r} + \psi\mathrm{a} + \frac{\mathrm{a}c^2}{30r^3}(-74+47\cos^2\gamma+67\cos^2\beta) + \frac{\mathrm{a}c^4}{70r^5}(-272+300\cos^2\gamma+356\cos^2\beta-123\cos^4\gamma-151\cos^4\beta - 110\cos^2\beta\cos^2\gamma)\right]. \quad (46).$$

If we put $\beta = \gamma = 0$, and note that c becomes $\frac{\mathrm{a}}{2^{1/3}}$, this expression reduces as before to the correct form.

§ 14. *Ellipsoidal Harmonic Deformations of the Third Order.*

The two ellipsoids whose forms have been determined are subject to further deformation by harmonic inequalities. The expression for the ellipticity f_i^s corresponding to all the cosine-harmonic functions for i greater than 2 and s even is given in (25), viz. :—

$$f_i^s = -\tfrac{1}{3}(2i+1)\frac{\mathfrak{Q}_i^s(r/k)\,\mathfrak{P}_i^s(\nu_0)\,\mathfrak{P}_i^s(1)\,\mathfrak{C}_i^s(\tfrac{1}{2}\pi)}{\lambda\mathfrak{T}_i^s\left\{\mathfrak{A}_i^s - \mathsf{A}_1^1 - \frac{k^3}{3\lambda r^3}\frac{\cos^2\gamma}{\sin^2\beta}\left[1+\frac{3(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2}+\ldots\right]\right\}} \quad . \quad (47).$$

I shall begin by considering the ellipticities f_3 and f_3^2 corresponding to $i = 3$ and $s = 0, 2$.

I define

$$\mathfrak{P}_3^s(\nu) = \nu(\kappa^2\nu - q^2), \quad (s = 0, 2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48),$$

where $q^2 = \frac{2}{5}[1+\kappa^2 \mp \surd(1-\frac{7}{4}\kappa^2+\kappa^4)]$, with upper sign for $s = 0$ and lower for $s = 2$.

Then

$$\mathfrak{Q}_3^s(\nu) = \mathfrak{P}_3^s(\nu)\int_\nu^\infty \frac{d\nu}{[\mathfrak{P}_3^s(\nu)]^2(\nu^2-1)^{1/2}(\nu^2-1/\kappa^2)^{1/2}}.$$

Since ν is always greater than unity, the function under the integral sign may be expanded in powers of $1/\nu$, as in (7), (8), (9) of § 5, and the integration may then be effected. In this way I find

$$\mathfrak{Q}_3^s(\nu) = \frac{1}{7\kappa^2\nu^4}\left\{1+\frac{7(1+\kappa^2)+10q^2}{2.9\kappa^2\nu^2} + \frac{7[9(3\kappa^4+2\kappa^2+3)+28q^2(\kappa^2+1)+40q^4]}{8.9.11\kappa^4\nu^4}+\ldots\right\} \quad (49).$$

In all the cases we have to consider κ^2 is nearly unity and κ'^2 is small. Then, since $q^2 = \frac{1}{5}[4-2\kappa'^2 \mp \surd(1-\kappa'^2+4\kappa'^4)]$, and since the function under the square root may be expanded in powers of κ'^2, we may obtain approximate expressions for q^2 in the two cases $s = 0$, $s = 2$.

When $s = 0$, we have

$$q^2 = \tfrac{3}{5}(1-\tfrac{1}{2}\kappa'^2-\tfrac{5}{8}\kappa'^4-\tfrac{5}{16}\kappa'^6\ldots).$$

When $s = 2$,

$$q^2 = 1 - \tfrac{1}{2}\kappa'^2 + \tfrac{3}{8}\kappa'^4 + \tfrac{3}{16}\kappa'^6 \ldots$$

If we substitute these values for q^2 in (49), and express the functions of κ^2 therein in terms of κ'^2, we obtain the following results:—

$$\mathfrak{Q}_3\left(\frac{k}{r}\right) = \frac{k^4}{7\kappa^2 r^4}\left\{1 + \tfrac{10}{9}\left(1 + \tfrac{1}{2}\kappa'^2 + \tfrac{5}{16}\kappa'^4 + \tfrac{7}{32}\kappa'^6 \ldots\right)\frac{k^2}{r^2} + \tfrac{35}{33}\left(1 + \kappa'^2 + \kappa'^4 + \kappa'^6 \ldots\right)\frac{k^4}{r^4} \ldots\right\}$$

$$\mathfrak{Q}_3^2\left(\frac{k}{r}\right) = \frac{k^4}{7\kappa^2 r^4}\left\{1 + \tfrac{4}{3}\left(1 + \tfrac{1}{2}\kappa'^2 + \tfrac{21}{32}\kappa'^4 + \tfrac{47}{64}\kappa'^6 \ldots\right)\frac{k^2}{r^2} + \tfrac{49}{33}\left(1 + \kappa'^2 + \tfrac{45}{28}\kappa'^4 + \tfrac{31}{14}\kappa'^6 \ldots\right)\frac{k^4}{r^4} \ldots\right\}.$$

Shorter forms may be given to these by making the expansions run in powers of $1/\kappa r^2$; we then have

$$\left.\begin{aligned}\mathfrak{Q}_3\left(\frac{k}{r}\right) &= \frac{k^4}{7\kappa^2 r^4}\left\{1 + \tfrac{10}{9}\left(1 + 0\kappa'^2 - \tfrac{1}{16}\kappa'^4 - \tfrac{1}{8}\kappa'^6 \ldots\right)\frac{k^2}{\kappa r^2} + \tfrac{35}{33}\frac{k^4}{\kappa^2 r^4} \ldots\right\}\\ \mathfrak{Q}_3^2\left(\frac{k}{r}\right) &= \frac{k^4}{7\kappa^2 r^4}\left\{1 + \tfrac{4}{3}\left(1 + 0\kappa'^2 + \tfrac{9}{32}\kappa'^4 + \tfrac{9}{32}\kappa'^6 \ldots\right)\frac{k^2}{\kappa r^2} + \tfrac{49}{33}\left(1 + 0\kappa'^2 + \tfrac{17}{28}\kappa'^4 + \tfrac{17}{28}\kappa'^6 \ldots\right)\frac{k^4}{\kappa^2 r^4} \ldots\right\}\end{aligned}\right\} \quad (50).$$

It will however suffice for our purposes to take

$$\left.\begin{aligned}\mathfrak{Q}_3\left(\frac{k}{r}\right) &= \frac{k^4}{7\kappa^2 r^4}\left\{1 + \tfrac{10}{9}\frac{k^2}{\kappa r^2}\right\}\\ \mathfrak{Q}_3^2\left(\frac{k}{r}\right) &= \frac{k^4}{7\kappa^2 r^4}\left\{1 + \tfrac{4}{3}\frac{k^2}{\kappa r^2}\right\}\end{aligned}\right\} \quad (51).$$

The next task is to determine the product $\mathfrak{P}_3^s(\nu_0)\,\mathfrak{P}_3^s(1)\,\mathsf{C}_3^s(\tfrac{1}{2}\pi) \quad (s = 0, 2)$.

The form of the ellipsoid is determined by ν_0, where $\nu_0 = \dfrac{1}{\kappa \sin\gamma} = \dfrac{1}{\sin\beta}$.

If we write $\Delta_1^2 = 1 - q^2 \sin^2\gamma$, with the definition of $\mathfrak{P}_3^s$ given above in (48), we have

$$\mathfrak{P}_3^s(\nu_0) = \frac{\Delta_1^2}{\kappa \sin^3\gamma} \quad (52).$$

It will be remembered that q has a different value according as $s = 0$ or 2.

I now make the following definitions,

$$\mathfrak{P}_3^s(\mu) = \mu(\kappa^2\mu^2 - q^2), \quad \mathsf{C}_3^s(\phi) = (q'^2 - \kappa'^2\cos^2\phi)\sqrt{(1 - \kappa'^2\cos^2\phi)};$$

so that

$$\mathfrak{P}_3^s(1)\,\mathsf{C}_3^s(\tfrac{1}{2}\pi) = q'^2(\kappa^2 - q^2) \quad (53).$$

It is easy to show that rigorously

$$q'^2(\kappa^2 - q^2) = \tfrac{2}{25}\left[1 - \kappa'^2 - \kappa'^4 \pm (1 - \tfrac{1}{2}\kappa'^2)\sqrt{(1 - \kappa'^2 + 4\kappa'^4)}\right] \quad (s = 0, 2).$$

Whence approximately, with the upper sign for $s = 0$,

$$q'^2(\kappa^2 - q^2) = \tfrac{4}{25}\left(1 - \kappa'^2 + \tfrac{9}{16}\kappa'^4 + 0\kappa'^6 \ldots\right) \quad (54).$$

With the lower sign for $s = 2$,

$$q'^2(\kappa^2 - q^2) = -\tfrac{1}{4}\kappa'^4(1 + 0\kappa'^2 - \tfrac{9}{16}\kappa'^4 \ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

The three last expressions (53), (54), (55), give the two values of $\mathfrak{P}_3^s(1)\,\mathsf{C}_3^s(\tfrac{1}{2}\pi)$. We now turn to the functions $\mathfrak{T}_3^s$. The general definition of § 5 was

$$\mathfrak{T}_i^s = \frac{(2i+1)\rho}{3M}\int[\mathfrak{P}_i^s(\mu)\,\mathfrak{C}_i^s(\phi)]^2 p\, d\sigma,$$

where M was the mass of the ellipsoid. Hence in the cases under consideration

$$\mathfrak{T}_3^s = \frac{7\sin^3\beta}{4\pi k^3\cos\beta\cos\gamma}\int[\mathfrak{P}_3^s(\mu)\,\mathsf{C}_3^s(\phi)]^2 p\, d\sigma \quad (s = 0, 2).$$

These integrals are evaluated in (38) of my paper on 'Integrals'; whence I find

$$\mathfrak{T}_3^s = \frac{2^3}{5^4}D[(1 - \tfrac{1}{2}\kappa'^2)(1 - \kappa'^2 - \tfrac{8}{3}\kappa'^4) + (1 - \kappa'^2 + \tfrac{2}{3}\kappa'^4)D],$$

where $D = \pm\sqrt{(1 - \kappa'^2 + 4\kappa'^4)}$, with upper sign for $s = 0$ and lower for $s = 2$.

It will, however, suffice if we use the development of D in powers of κ'^2. The result is, in fact, given in the equations next below (38) in 'Integrals,' and they are

$$\left.\begin{aligned} \mathfrak{T}_3 &= (\tfrac{2}{5})^4(1 - 2\kappa'^2 + \tfrac{49}{16}\kappa'^4 \ldots), \\ \mathfrak{T}_3^2 &= \tfrac{1}{3.5}\kappa'^4(1 - \kappa'^2 + \tfrac{31}{16}\kappa'^4 \ldots). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56).$$

From (53), (54), (55), and (56) we now find

$$\left.\begin{aligned} \frac{\mathfrak{P}_3(1)\,\mathsf{C}_3(\tfrac{1}{2}\pi)}{\mathfrak{T}_3} &= \tfrac{25}{4}\frac{1 - \kappa'^2 + \tfrac{9}{16}\kappa'^4 \ldots}{1 - 2\kappa'^2 + \tfrac{49}{16}\kappa'^4 \ldots} = \tfrac{25}{4}(1 + \kappa'^2 - \tfrac{1}{2}\kappa'^4 \ldots), \\ \frac{\mathfrak{P}_3^2(1)\,\mathsf{C}_3^2(\tfrac{1}{2}\pi)}{\mathfrak{T}_3^2} &= -\tfrac{15}{4}\frac{1 + 0\kappa'^2 - \tfrac{9}{16}\kappa'^4 \ldots}{1 - \kappa'^2 + \tfrac{31}{16}\kappa'^4 \ldots} = -\tfrac{15}{4}(1 + \kappa'^2 - \tfrac{3}{2}\kappa'^4 \ldots). \end{aligned}\right\} \quad . \quad (57).$$

Thus from (51), (52), (57), we find

$$\left.\begin{aligned} -\frac{7}{3\lambda\mathfrak{T}_3}\mathfrak{Q}_3\left(\frac{r}{k}\right)\mathfrak{P}_3(\nu_0)\,\mathfrak{P}_3(1)\,\mathsf{C}_3(\tfrac{1}{2}\pi) &= -\frac{25\Delta_1^2}{12\lambda\sin^3\beta}(1 + \kappa'^2 - \tfrac{1}{2}\kappa'^4 \ldots)\frac{k^4}{r^4}\left(1 + \tfrac{10}{9}\frac{k^2}{\kappa r^2}\right), \\ -\frac{7}{3\lambda\mathfrak{T}_3^2}\mathfrak{Q}_3^2\left(\frac{r}{k}\right)\mathfrak{P}_3^2(\nu_0)\,\mathfrak{P}_3^2(1)\,\mathsf{C}_3^2(\tfrac{1}{2}\pi) &= \frac{5\Delta_1^2}{4\lambda\sin^3\beta}(1 + \kappa'^2 - \tfrac{3}{2}\kappa'^4)\frac{k^4}{r^4}\left(1 + \tfrac{4}{3}\frac{k^2}{\kappa r^2}\right). \end{aligned}\right\} (58).$$

By (47) f_3^s is equal to the above expressions divided by

$$\mathfrak{A}_3^s - \mathbf{A}_1^1 - \frac{k^3}{3\lambda r^3}\frac{\cos^2\gamma}{\sin^2\beta}\left[1 + \frac{3(3+\kappa^2)}{14\kappa^2}\frac{k^2}{r^2}\ldots\right].$$

It therefore remains to determine $\mathfrak{A}_3^s$ and $\mathbf{A}_1^1$.

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In order to find A_1^1 we expand Δ in powers of $\kappa'^2 \tan^2\gamma$. Thus

$$\mathsf{A}_1^1 = \kappa\cot^2\gamma\int_0^\gamma \frac{\tan^2\gamma}{\Delta}\,d\gamma = \kappa\cot^2\gamma\int_0^\gamma \frac{\sin^2\gamma}{\cos^3\gamma}\left(1-\tfrac{1}{2}\kappa'^2\tan^2\gamma+\tfrac{3}{8}\kappa'^4\tan^4\gamma\ldots\right)d\gamma\,;$$

whence

$$\mathsf{A}_1^1 = \frac{\kappa}{2\sin\gamma}\{[1+\tfrac{3}{8}\kappa'^2+\tfrac{15}{64}\kappa'^4][1-\cos^2\gamma\Omega]-\tfrac{1}{4}\kappa'^2\tan^2\gamma\,(1+\tfrac{5}{8}\kappa'^2)+\tfrac{1}{8}\kappa'^4\tan^4\gamma\}.$$

The result may, of course, be obtained with higher accuracy if it be desired to do so.

By (25) of the 'Pear-shaped Figure'

$$\mathfrak{A}_3^s = \mathfrak{P}_3^s(\nu_0)\,\mathfrak{Q}_3^s(\nu_0) = \frac{\kappa\Delta_1^4}{\sin^6\gamma}\int_0^\gamma \frac{\sin^6\gamma}{\Delta_1^4\,\Delta}\,d\gamma,$$

where $\Delta_1^2 = 1-q^2\sin^2\gamma$, with appropriate values of q for $s = 0$ and 2 respectively.

It is clear that under the integral sign $1/\Delta$ may be expanded in powers of $\kappa'^2\tan^2\gamma$, but this is not possible with $1/\Delta_1^4$.

The two cases $s = 0$, 2 have to be treated by different methods, and I begin with $\mathfrak{A}_3$, where $s = 0$.

Since

$$\frac{1}{\Delta} = \frac{1}{\cos\gamma}[1+\tfrac{1}{2}\kappa'^2\tan^2\gamma+\tfrac{3}{8}\kappa'^4\tan^4\gamma\ldots],$$

it is necessary to consider the integrals with respect to γ of the functions

$$\frac{\sin^6\gamma}{\cos\gamma\,\Delta_1^4},\quad \frac{\sin^8\gamma}{\cos^3\gamma\,\Delta_1^4},\quad \frac{\sin^{10}\gamma}{\cos^5\gamma\,\Delta_1^4}.$$

Writing $x = \sin\gamma$, and $q^2 = \dfrac{1}{a^2}$, this is seen to be equivalent to finding

$$\int\frac{x^6\,dx}{(1-x^2)(a^2-x^2)^2},\quad \int\frac{x^8\,dx}{(1-x^2)^2(a^2-x^2)^2},\quad \int\frac{x^{10}\,dx}{(1-x^2)^3(a^2-x^2)^2}.$$

Now

$$\frac{x^6}{(1-x^2)(a^2-x^2)^2} = -1+\frac{1}{2(a^2-1)^2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)-\frac{a^4}{4(a^2-1)}\left(\frac{1}{(a-x)^2}+\frac{1}{(a+x)^2}\right)$$
$$+\frac{a^3(3a^2-5)}{4(a^2-1)^2}\left(\frac{1}{a-x}+\frac{1}{a+x}\right),$$

$$\frac{x^8}{(1-x^2)^2(a^2-x^2)^2} = 1-\frac{7a^2-3}{4(a^2-1)^3}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)+\frac{1}{4(a^2-1)^2}\left(\frac{1}{(1-x)^2}+\frac{1}{(1+x)^2}\right)$$
$$-\frac{a^5(3a^2-7)}{4(a^2-1)^3}\left(\frac{1}{a-x}+\frac{1}{a+x}\right)+\frac{a^6}{4(a^2-1)^2}\left(\frac{1}{(a-x)^2}+\frac{1}{(a+x)^2}\right),$$

$$\frac{x^{10}}{(1-x^2)^3(a^2-x^2)^2} = -1+\frac{3(21a^4-18a^2+5)}{16(a^2-1)^4}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)-\frac{17a^2-9}{16(a^2-1)^3}\left(\frac{1}{(1-x)^2}+\frac{1}{(1+x)^2}\right)$$
$$+\frac{1}{8(a^2-1)^2}\left(\frac{1}{(1-x)^3}+\frac{1}{(1+x)^3}\right)+\frac{3a^7(a^2-3)}{4(a^2-1)^4}\left(\frac{1}{a-x}+\frac{1}{a+x}\right)$$
$$-\frac{a^8}{4(a^2-1)^3}\left(\frac{1}{(a-x)^2}+\frac{1}{(a+x)^2}\right).$$

Each term in these expressions is integrable, and the limits of integration are $x = \sin\gamma$ to 0. After integration a must be put equal to $1/q$, and since the result will only be true as far as κ'^4 we must put for q^2 its approximate value $\frac{3}{5}(1-\frac{1}{2}\kappa'^2-\frac{5}{8}\kappa'^4)$.

When these processes are carried out, and the formulæ combined, we find

$$\mathfrak{A}_3 = \frac{\Delta_1^4}{q^4\sin^5\gamma}\left\{-1+\tfrac{9}{4}(1+\tfrac{1}{4}\kappa'^2+\tfrac{23}{64}\kappa'^4)\,\Omega-\tfrac{5}{4}(1+0\kappa'^2-\tfrac{9}{32}\kappa'^4)\frac{1}{\Delta_1^2}\right.$$
$$\left.-\tfrac{9}{16}\kappa'^2\sec^2\gamma\,(1+\tfrac{39}{16}\kappa'^2)+\tfrac{27}{128}\kappa'^4\sec^4\gamma\right\} \quad . \quad . \quad . \quad (59).$$

In the coefficient and in Δ_1^2 (or $1-q^2\sin^2\gamma$) it is as easy to use the rigorous value of q^2 as the approximate one, so it may be well to repeat that

$$q^2 = \tfrac{2}{5}[1+\kappa^2-\sqrt{(1-\tfrac{7}{4}\kappa^2+\kappa^4)}], \text{ and } \Omega = \frac{1}{\sin\gamma}\log_e\frac{1+\sin\gamma}{\cos\gamma}.$$

If a process parallel to that adopted for finding $\mathfrak{A}_3$ were adopted in the case of $\mathfrak{A}_3^2$, it would lead to a divergent series, but fortunately a much simpler process is available.

In the case of $i = 3$, $s = 2$, we have $q^2 = 1-\frac{1}{2}\kappa'^2+\frac{3}{8}\kappa'^4\ldots$, so that

$$\Delta_1^2 = \cos^2\gamma\,[1+\tfrac{1}{2}\kappa'^2(1-\tfrac{3}{4}\kappa'^2)\tan^2\gamma].$$

Accordingly we are now able to expand $1/\Delta_1^4$ in powers of $\kappa'^2\tan^2\gamma$.

We have then

$$\frac{1}{\Delta_1^4} = \frac{1}{\cos^4\gamma}[1-\kappa'^2(1-\tfrac{3}{4}\kappa'^2)\tan^2\gamma+\tfrac{3}{4}\kappa'^4\tan^4\gamma\ldots],$$

$$\frac{1}{\Delta} = \frac{1}{\cos\gamma}[1-\tfrac{1}{2}\kappa'^2\tan^2\gamma+\tfrac{3}{8}\kappa'^4\tan^4\gamma\ldots].$$

Whence

$$\frac{\sin^6\gamma}{\Delta\Delta_1^4} = \frac{\sin^6\gamma}{\cos^5\gamma}[1-\tfrac{3}{2}\kappa'^2(1-\tfrac{1}{2}\kappa'^2)\tan^2\gamma+\tfrac{13}{8}\kappa'^4\tan^4\gamma\ldots].$$

Now

$$\int_0^\gamma\frac{\sin^6\gamma}{\cos^5\gamma}d\gamma = \sin\gamma\,[\tfrac{1}{4}\tan^4\gamma-\tfrac{5}{4\cdot2}\tan^2\gamma+\tfrac{5\cdot3}{4\cdot2}(\Omega-1)],$$

$$\int_0^\gamma\frac{\sin^8\gamma}{\cos^7\gamma}d\gamma = \sin\gamma\,[\tfrac{1}{6}\tan^6\gamma-\tfrac{7}{6\cdot4}\tan^4\gamma+\tfrac{7\cdot5}{6\cdot4\cdot2}\tan^2\gamma-\tfrac{7\cdot5\cdot3}{6\cdot4\cdot2}(\Omega-1)],$$

$$\int_0^\gamma\frac{\sin^{10}\gamma}{\cos^9\gamma}d\gamma = \sin\gamma\,[\tfrac{1}{8}\tan^8\gamma-\tfrac{9}{8\cdot6}\tan^6\gamma+\tfrac{9\cdot7}{8\cdot6\cdot4}\tan^4\gamma-\tfrac{9\cdot7\cdot5}{8\cdot6\cdot4\cdot2}\tan^2\gamma+\tfrac{9\cdot7\cdot5\cdot3}{8\cdot6\cdot4\cdot2}(\Omega-1)],$$

where

$$\Omega = \frac{1}{\sin\gamma}\log_e\frac{1+\sin\gamma}{\cos\gamma}.$$

Combining these formulæ, I find

$$\mathfrak{A}_3^2 = \frac{\Delta_1^4}{4\sin^5\gamma}\left[\left\{\tan^4\gamma - \tfrac{5}{2}\tan^2\gamma + \tfrac{15}{2}(\Omega - 1)\right\}\left\{1 + \tfrac{5}{4}\kappa'^2 + \tfrac{33}{128}\kappa'^4\right\}\right.$$
$$\left. - \kappa'^2\left(1 + \tfrac{7}{32}\kappa'^2\right)\tan^6\gamma + \tfrac{13}{16}\kappa'^4\tan^8\gamma\right] \quad . \quad . \quad (60).$$

The form is somewhat awkward, but I have not been able to reduce it to any better shape.

§ 15. *The Values of $\mathfrak{Q}_i^s$ for Higher Harmonic Terms.*

For higher harmonic terms it is necessary to adopt the approximate forms of the functions investigated in "Harmonics." The development is there carried out in powers of a parameter β, which I will now write β_0 to avoid confusion; this parameter is equal to $\frac{1-\kappa^2}{1+\kappa^2}$, or to $\frac{1}{2}\kappa'^2 + \frac{1}{4}\kappa'^4 \ldots$ of the present paper.

The functions are here defined by

$$\mathfrak{Q}_i^s(\nu) = \mathfrak{P}_i^s(\nu)\int_\nu^\infty \frac{d\nu}{[\mathfrak{P}_i^s(\nu)]^2(\nu^2-1)^{1/2}\left(\nu^2 - \frac{1+\beta_0}{1-\beta_0}\right)^{1/2}};$$

but in the notation of § 10 of "Harmonics" this would be called $\mathfrak{Q}_i^s(\nu)/\mathfrak{E}_i^s$. Thus, if $[\mathfrak{Q}_i^s(\nu)]$ denotes that function as defined in "Harmonics," we have

$$\mathfrak{Q}_i^s(\nu) = \frac{[\mathfrak{Q}_i^s(\nu)]}{\mathfrak{E}_i^s}.$$

We have for the approximate expression for $\mathfrak{P}_i^s$

$$\mathfrak{P}_i^s(\nu) = P_i^s(\nu) + \beta_0 q_{s-2} P_i^{s-2}(\nu) + \beta_0 q_{s+2} P_i^{s+2}(\nu) + \beta_0^2 q_{s-4} P_i^{s-4}(\nu) + \beta_0^2 q_{s+4} P_i^{s+4}(\nu).$$

The investigation on p. 500 of "Harmonics" shows that the leading term of $[\mathfrak{Q}_i^s(\nu)]$ is

$$(-)^s \frac{2^i\, i\,!}{2i+1\,!}\,\frac{i+s\,!}{\nu^{i+1}}\left[1 + \beta_0 q_{s-2}\frac{i+s-2\,!}{i+s\,!} + \beta_0 q_{s+2}\frac{i+s+2\,!}{i+s\,!}\right.$$
$$\left. + \beta_0^2 q_{s-4}\frac{i+s-4\,!}{i+s\,!} + \beta_0^2 q_{s+4}\frac{i+s+4\,!}{i+s\,!}\right].$$

This has to be divided by $\mathfrak{E}_i^s$, the formula for which is given in § 10 of "Harmonics," and we thus obtain the leading term of $\mathfrak{Q}_i^s(\nu)$.

For the second term it will suffice if we take β_0 as zero, so that it is only necessary to consider $Q_i^s(\nu)$, which is equal to $(\nu^2-1)^{s/2}\frac{d^s}{d\nu^s}Q_i(\nu)$.

Since

$$Q_i(\nu) = \frac{2^i (i\,!)^2}{2i+1\,!}\left[\frac{1}{\nu^{i+1}} + \frac{i+2\,!}{2.1\,!\,i\,!}\frac{1}{(2i+3)\nu^{i+3}} + \ldots\right],$$

by differentiation, and by expansion of $(\nu^2-1)^{s/2}$ in powers of $1/\nu^2$ we obtain

$$Q_i^s(\nu) = (-)^s \frac{2^i.i\,!\,.\,i+s\,!}{2i+1\,!\,\nu^{i+1}}\left[1 + \frac{(i+2)(i+1)+s^2}{2(2i+3)\nu^2} + \ldots\right].$$

Accordingly, in order to find the second term to the degree of approximation adopted, it is merely necessary to multiply the leading term by

$$1 + \frac{(i+2)(i+1)+s^2}{2(2i+3)\nu^2}.$$

In order to find the leading term explicitly we have to insert for the q's their values, and after some tedious reductions I find

$$\mathfrak{Q}_i^s\left(\frac{r}{k}\right) = \frac{2^i.i\,!\,i-s\,!}{2i+1\,!}\frac{k^{i+1}}{r^{i+1}}\{1+\tfrac{1}{4}\beta_0(\Sigma+2i)+\tfrac{1}{64}\beta_0^2[-2\Upsilon+\Sigma^2(s^2+3)$$
$$+2\Sigma(6i-1+2s^2)+2i(3i+1)-s^2]\}\times\left\{1+\frac{(i+2)(i+1)+s^2}{2(2i+3)}\frac{k^2}{r^2}\right\} \quad . \quad . \quad (61),$$

where

$$\Sigma = \frac{i(i+1)}{s^2-1}, \quad \Upsilon = \frac{(i-1)\,i\,(i+1)(i+2)}{s^2-4}.$$

This formula fails for the cases of $s = 0$ and $s = 2$, and these cases have to be treated apart. Following a parallel procedure I find

$$\mathfrak{Q}_i^2\left(\frac{r}{k}\right) = \frac{2^i.i\,!\,i-2\,!}{2i+1\,!}\frac{k^{i+1}}{r^{i+1}}\{1+\tfrac{1}{4}\beta_0(\Sigma+2i)+\tfrac{1}{512}\beta_0^2[29\Sigma^2+(144i+298)\Sigma-48i-40]\}$$
$$\times\left\{1+\frac{(i+2)(i+1)+4}{2(2i+3)}\frac{k^2}{r^2}\right\}. \quad (62).$$

$$\mathfrak{Q}_i\left(\frac{r}{k}\right)=\frac{2^i(i!)^2}{2i+1!}\frac{k^{i+1}}{r^{i+1}}\{1-\tfrac{1}{4}\beta_0 i(i-1)+\tfrac{1}{128}\beta_0^2 i(i-1)(7i^2-3i-6)\}\left\{1+\frac{(i+2)(i+1)}{2(2i+3)}\frac{k^2}{r^2}\right\}(63).$$

The values for i less than 3 are not required, and when $i = 3$ these formulæ are found to agree *mutatis mutandis* with those of the last section.

It is pretty clear from general considerations that the higher inequalities corresponding to harmonics other than the zonal ones must be very small. I have, in fact, computed the third tesseral harmonic inequality ($i = 3$, $s = 2$), and find that it is so very minute compared with the third zonal inequality ($i = 3$, $s = 0$) as to be negligible. Accordingly it appeared to be a waste of time to develop formulæ for any other than zonal inequalities for values of i greater than 3. Thus of the formulæ just determined the only one of which actual use is made is (63).

§ 16. *The Fourth Zonal Harmonic Inequality.*

In developing the expressions for the higher harmonic inequalities it seems to be most convenient to retain the parameter β_0, which is equal to $\frac{1-\kappa^2}{1+\kappa^2}$, instead of developing in powers of κ'^2 as heretofore.

On putting $i = 4$ in (63), we find

$$\mathfrak{Q}_4\left(\frac{r}{k}\right) = \frac{8}{5\cdot 7\cdot 9}\frac{k^5}{r^5}\left(1+\tfrac{15}{11}\frac{k^2}{r^2}\right)(1-3\beta_0+\tfrac{141}{16}\beta_0^2).$$

With the notation of "Harmonics" we have

$$\mathfrak{P}_4(\nu) = P_4(\nu)+\tfrac{1}{4}\beta_0 P_4^2(\nu)+\tfrac{1}{128}\beta_0^2 P_4^4(\nu),$$
$$\mathfrak{C}_4(\phi) = 1-5\beta_0\cos 2\phi+\tfrac{35}{16}\beta_0^2\cos 4\phi.$$

Accordingly

$$\mathfrak{P}_4(1) = P_4(1) = 1,$$
$$\mathfrak{C}_4(\tfrac{1}{2}\pi) = 1+5\beta_0+\tfrac{35}{16}\beta_0^2.$$

Again, from § 22 of "Harmonics" for type EEC, $i = 4$, $s = 0$, we have

$$\int[\mathfrak{P}_4\mathfrak{C}_4]^2 p\,d\sigma = \tfrac{4}{9}\pi k^3\nu_0(\nu_0^2-1)^{1/2}\left(\nu_0^2-\frac{1+\beta_0}{1-\beta_0}\right)^{1/2}[1+10\beta_0+\tfrac{325}{8}\beta_0^2].$$

But $\mathfrak{T}_4$ is this integral multiplied by 9 and divided by 3 times the volume of the ellipsoid.

Therefore

$$\mathfrak{T}_4 = 1+10\beta_0+\tfrac{325}{8}\beta_0^2.$$

In this formula the coefficients of the powers of β_0 increase with great rapidity, and the approximation may not be very satisfactory; nevertheless it is the best attainable without an enormous increase of labour.

Combining our several results

$$-\frac{9}{3\lambda\mathfrak{T}_4}\mathfrak{Q}_4\left(\frac{r}{k}\right)\mathfrak{P}_4(1)\,\mathfrak{C}_4(\tfrac{1}{2}\pi) = -\frac{8}{105\lambda}\frac{k^5}{r^5}\left(1+\tfrac{15}{11}\frac{k^2}{r^2}\right)(1-8\beta_0+\tfrac{283}{8}\beta_0^2) \quad . \quad (64).$$

It remains to find $\mathfrak{P}_4(\nu_0)$, and the denominator in the expression for f_4 which involves $\mathfrak{A}_4-\mathsf{A}_1^1$.

We have

$$\mathfrak{A}_4 = \mathfrak{P}_4(\nu_0)\,\mathfrak{Q}_4(\nu_0)$$
$$= [\mathfrak{P}_4(\nu_0)]^2\int_{\nu_0}^{\infty}\frac{d\nu}{[\mathfrak{P}_4(\nu)]^2(\nu^2-1)^{1/2}(\nu^2-1/\kappa^2)^{1/2}}.$$

Inside the integral sign I write $\nu = \frac{1}{\kappa\sin\psi}$ and change the independent variable to ψ; I also put $\sin\chi = \kappa \sin\psi$. At the surface of the ellipsoid we have $\psi = \gamma$, $\chi = \beta$, and since $\nu_0 = \dfrac{1}{\kappa \sin\gamma} = \dfrac{1}{\sin\beta}$, these are the values to be used in $\mathfrak{P}_4(\nu_0)$ outside the integral sign.

Then we have

$$\int_{\nu_0}^{\infty} \frac{d\nu}{(\nu^2-1)^{1/2}(\nu^2-1/\kappa^2)^{1/2}} = \kappa \int_0^{\gamma} \sec\chi \, d\psi.$$

Since

$$P_4(\nu) = \tfrac{35}{8}(\nu^2-1)^4 + 5(\nu^2-1) + 1 = 1 + 5\cot^2\chi + \tfrac{35}{8}\cot^4\chi,$$
$$P_4^2(\nu) = \tfrac{15}{2}(\nu^2-1)[7(\nu^2-1)+6] = \tfrac{15}{2}(6\cot^2\chi + 7\cot^4\chi),$$
$$P_4^4(\nu) = 105(\nu^2-1)^2 \qquad = 105\cot^4\chi.$$

Since further

$$\mathfrak{P}_4(\nu) = P_4(\nu) + \tfrac{1}{4}\beta_0 P_4^2(\nu) + \tfrac{1}{128}\beta_0^2 P_4^4(\nu),$$

we find

$$\left.\begin{aligned} \mathfrak{P}_4(\nu) &= 1 + a\cot^2\chi + b\cot^4\chi, \\ a &= 5(1+\tfrac{9}{4}\beta_0), \\ b &= \tfrac{35}{8}(1+3\beta_0+\tfrac{3}{16}\beta_0^2). \end{aligned}\right\} \quad \ldots \ldots \quad (65)$$

where

It must be noted further that when $\nu = \nu_0$ at the surface of the ellipsoid, $\chi = \beta$.

It follows then that $\mathfrak{P}_4(\nu_0) = 1 + a\cot^2\beta + b\cot^4\beta$; and from (64)

$$-\frac{9}{3\lambda\mathfrak{T}_4}\mathfrak{Q}_4\left(\frac{r}{k}\right)\mathfrak{P}_4(\nu_0)\mathfrak{P}_4(1)\mathfrak{C}_4(\tfrac{1}{2}\pi) = -\frac{8}{105\lambda}\frac{k^5}{r^5}\left(1+\tfrac{15}{11}\frac{k^2}{r^2}\right)$$
$$(1-8\beta_0+\tfrac{283}{8}\beta_0^2)(1+a\cot^2\beta+b\cot^4\beta).$$

It remains to consider the evaluation of $\mathfrak{A}_4$, which now assumes the form

$$\mathfrak{A}_4 = \kappa \int_0^{\gamma} \left(\frac{1+a\cot^2\beta+b\cot^2\beta}{1+a\cot^2\chi+b\cot^2\chi}\right)^2 \sec\chi \, d\psi.$$

It would no doubt be possible to split the subject of integration into partial fractions, and thus obtain an accurate value as was done in the case of $\mathfrak{A}_3$, but it does not seem worth while to undertake so heavy a task, because a sufficiently exact value may be obtained by quadratures.

The method is employed in § 18, p. 294, in my paper on "Stability," and may be explained very shortly.

I divide γ into 10 or 12 equal parts—say 10 for brevity—and let $\delta = \frac{1}{10}\gamma$.

I then compute eleven equidistant values of the subject of integration, say u_0, u_1, ... u_{10}, corresponding to $\psi = 0, 2\delta, 3\delta, \ldots 10\delta$. As a fact it is unnecessary to compute the first four of these, because they are practically zero.

The equidistant values increase so rapidly that they are very inappropriate for the application of the rules of numerical quadratures. Accordingly I take an empirical and integrable function, say v, such that $v_{12} = u_{12}$ and $v_{11} = u_{11}$, and apply the rules of quadrature only to the differences $u_n - v_n$. The result is a correction to the integral $\int_0^\gamma v\, d\psi$.

The empirical function which satisfies these conditions is

$$v = u_{10} e^{\frac{\psi-\gamma}{\delta} \log_e \frac{u_{10}}{u_9}}.$$

When $\psi = \gamma = 10\delta$, $v = u_{10}$; and when $\psi = 9\delta$, $v = u_{10} e^{-\log_e \frac{u_{10}}{u_9}} = u_9$.

Then $\int_0^\gamma v\, d\psi = \dfrac{u_{10}\delta}{\log_e (u_{10}/u_9)} (1 - e^{-10 \log_e (u_{10}/u_9)})$.

In the cases we have to treat $e^{-10 \log_e (u_{10}/u_9)}$ is an extremely small fraction, so that practically $\int_0^\gamma v\, d\psi = \dfrac{u_{10}\gamma}{10 \log_e (u_{10}/u_9)}$; and this is the function to be corrected by the result of quadrature.

For the quadratures we have

$$v_{10} = u_{10}, \quad v_9 = u_9, \quad v_8 = u_{10}\left(\frac{u_9}{u_{10}}\right)^2, \quad v_7 = u_{10}\left(\frac{u_9}{u_{10}}\right)^3, \text{ \&c.}$$

Thus the equidistant values of the function to be integrated (arranged backwards) are

$$0, \quad 0, \quad u_8 - u_{10}\left(\frac{u_9}{u_{10}}\right)^2, \quad u_7 - u_{10}\left(\frac{u_9}{u_{10}}\right)^3, \text{ \&c.}$$

The first two are zero, the next three or four are sensible, and the rest are insensible; thus the quadrature is very short. The correction is found to be very small, and we might perhaps have been content with the empirical integral without material loss of accuracy.

§ 17. *The Fifth Zonal Harmonic Inequality.*

This is treated exactly in the same way as the fourth, and I will only give the results.

We have

$$\mathfrak{Q}_5\left(\frac{r}{k}\right) = \frac{8}{7 \cdot 9 \cdot 11} \frac{k^6}{r^6}\left(1 + \tfrac{21}{13}\frac{k^2}{r^2}\right)(1 - 5\beta_0 + \tfrac{385}{16}\beta_0^2),$$

$$\mathfrak{P}_5(\nu) = P_5(\nu) + \tfrac{1}{4}\beta_0 P_5^2(\nu) + \tfrac{1}{128}\beta_0^2 P_5^4(\nu),$$

$$\mathsf{C}_5(\nu) = \sqrt{(1-\beta_0 \cos 2\phi)}\,.\,[1 - 7\beta_0 \cos 2\phi + \tfrac{63}{16}\beta_0^2 \cos 4\phi],$$

$$\mathfrak{T}_5 = 1 + 15\beta_0 + \tfrac{777}{8}\beta_0^2.$$

Whence

$$-\frac{11}{3\lambda\mathfrak{T}_5}\mathfrak{Q}_5\left(\frac{r}{k}\right)\mathfrak{P}_5(1)\,\mathsf{C}_5(\tfrac{1}{2}\pi) = -\frac{8}{189\lambda}\frac{k^6}{r^6}\left(1 + \tfrac{21}{13}\frac{k^2}{r^2}\right)(1 - 13\beta_0 + \tfrac{727}{8}\beta_0^2)\sqrt{(1+\beta_0)}.$$

Then

$$\mathfrak{P}_5(\nu_0) = \operatorname{cosec}\beta\,(1+a\cot^2\beta+b\cot^2\beta),$$

where

$$a = 7\,(1+\tfrac{15}{4}\beta_0),$$

$$b = \tfrac{63}{8}\,(1+5\beta_0+\tfrac{15}{128}\beta_0{}^2).$$

Finally

$$\mathfrak{A}_5 = \kappa\int_0^\gamma\left[\frac{\operatorname{cosec}\beta\,(1+a\cot^2\beta+b\cot^4\beta)}{\operatorname{cosec}\chi\,(1+a\cot^2\chi+b\cot^4\chi)}\right]^2\sec\chi\,d\psi,$$

which is to be evaluated by quadratures as was proposed for the fourth harmonic.

§ 18. *Moment of Momentum and Limiting Stability.*

The moment of momentum of the system is $I\omega$, but when we are determining the configuration of minimum moment of momentum, which is a figure of bifurcation and gives us the configuration of limiting stability, the conditions are different according as whether we are treating the problem of the figures of equilibrium where both masses are liquid, or ROCHE's problem in which the ellipsoid denoted by capital letters is rigid.

Accordingly I write

$$I = \tfrac{4}{3}\pi\rho\mathrm{a}^3\left[\tfrac{1}{5}\,\frac{\lambda}{1+\lambda}\,(b^2+c^2)+\frac{\lambda r^2}{(1+\lambda)^2}\right]+\tfrac{4}{3}\pi\rho\mathrm{a}^3\,.\,\tfrac{1}{5}\,\frac{1}{1+\lambda}(B^2+C^2),$$

and for determining the angular momentum of the figures of equilibrium I take the whole expression for I, but for ROCHE's problem omit the last term.

Since $\omega^2 = \tfrac{4}{3}\pi\rho\dfrac{\mathrm{a}^3}{r^3}(1+\zeta)$, I compute for ROCHE's problem

$$\mu_1 = \frac{\mathrm{a}^{3/2}}{r^{3/2}}(1+\zeta)^{1/2}\left[\tfrac{1}{5}\,\frac{\lambda}{1+\lambda}\,\frac{(b^2+c^2)}{\mathrm{a}^2}+\frac{\lambda r^2}{(1+\lambda)^2\,\mathrm{a}^2}\right];$$

and for the figures of equilibrium

$$\mu_2 = \mu_1+\frac{\mathrm{a}^{3/2}}{r^{3/2}}(1+\zeta)^{1/2}\,\frac{1}{5\,(1+\lambda)}\cdot\frac{B^2+C^2}{\mathrm{a}^2}.$$

The moment of momentum is given by

$$I\omega = (\tfrac{4}{3}\pi\rho)^{3/2}\,\mathrm{a}^5\,(\mu_1 \text{ or } \mu_2).$$

It will be observed that μ_1 and μ_2 are expressible by numbers for any given solution of the problem.

Suppose now that we have a succession of solutions for equidistant values of γ differing but little from one another. Then if the solutions lie close to the region of limiting stability, we shall find that one of them corresponds to minimum moment of momentum, either of μ_1 or of μ_2, as the case may be. Such a solution is a figure of bifurcation, and of the two coalescent solutions one has one more degree of instability than the other. If one of the two is continuous with a stable solution, and if, moreover, in the passage to the undoubtedly stable solution it passes through no other point of bifurcation, one of our two solutions is secularly stable and the other unstable.

Now, two liquid masses revolving about one another orbitally at an infinite distance are undoubtedly stable, and such a case is also continuous with one of our solutions. Further, SCHWARZSCHILD has proved that ROCHE'S ellipsoid has no point of bifurcation from first to last, and as this is true of one such ellipsoid, it is true of two.* Hence we conclude that the minima of μ_1 and μ_2 will afford figures of limiting stability.

§ 19. *Approximate Solution of the Problem.*

It is clear that spherical harmonic analysis is applicable to the case when the two liquid masses are widely distant. When they are so much deformed by their interaction that that method becomes inapplicable, good results may be obtained from the formulæ of the last sections by means of development in powers of sin γ, and it is this plan which is especially considered in the present section.

It appears from § 1 that when one of the masses is small compared with the other (λ small), the configuration of limiting stability for the problem of figures of equilibrium occurs when the two masses are very far apart. As λ increases, that configuration corresponds with diminishing radius vector. It seemed then probable that at least some of the solutions might be found by means of these series, and if this were so it might, in many cases, prove unnecessary to follow the same laborious procedure as in finding the limiting stability of ROCHE'S ellipsoid. This view was found to be correct, and I therefore think it well to record the methods by which the developments may be obtained, without however giving the full details of the very laborious analysis.

When the masses are far apart, the terms denoted ϵ and η in the equation for κ'^2 and in that for a^3/r^3 are small, and they must be neglected in the developments.

Writing for brevity $g = \sin\gamma$, we may prove that

$$\Omega = \frac{1}{\sin\gamma}\log_e\frac{1+\sin\gamma}{\cos\gamma} = \sum_0^\infty \frac{g^{2n}}{2n+1}, \quad \tan^2\gamma = -1+\sum_0^\infty g^{2n}, \quad \tan^4\gamma = 1+\sum_0^\infty (n-1)\,g^{2n},$$

$$\tan^6\gamma = -1+\sum_0^\infty \frac{(n-1)(n-2)}{1.2}g^{2n}, \quad \tan^8\gamma = 1+\sum_0^\infty \frac{(n-1)(n-2)(n-3)}{1.2.3}g^{2n}, \text{ \&c.}$$

Hence the developments may be obtained of the functions $\sigma_0, \sigma_1, \sigma_2 \ldots \tau_0, \tau_1, \tau_2 \ldots$,

* SCHWARZSCHILD, "Die Poincarésche Theorie des Gleichgewichts." 'Neue Annalen der k. Sternwarte München,' Band III, 1896.

and thence of $v_0, v_1, v_2 \dots$ in series proceeding by powers of g^2; and thence we may find κ'^2 in that form.

The result as far as g^4 is

$$\kappa'^2 = \frac{1+\lambda}{4+\lambda}\left[1 - \frac{30}{7(1+\lambda)}g^2 - \frac{12(11+26\lambda)}{7^2(1+\lambda)^2}g^4 \dots\right].$$

I have also found the term in g^6, but shall make no use of it.

With this value of κ'^2 or of κ^2, which is $1-\kappa'^2$, we develop the expression for a^3/r^3 in the same manner. The result is :—

$$\frac{\mathrm{a}^3}{r^3} = \tfrac{2}{5}\frac{1+\lambda}{4+\lambda}g^2\left[1 + \frac{5+\lambda}{7(4+\lambda)}g^2 - \frac{88+53\lambda}{7^2(4+\lambda)^2}g^4 \dots\right].$$

By inversion we have

$$g^2 = \sin^2\gamma = \frac{5(4+\lambda)}{2(1+\lambda)}\frac{\mathrm{a}^3}{r^3} - \frac{25(5+\lambda)(4+\lambda)}{2^2.7(1+\lambda)^2}\frac{\mathrm{a}^6}{r^6} + \frac{125(69+2\lambda)(2+\lambda)(4+\lambda)}{2^3.7^2(1+\lambda)^3}\frac{\mathrm{a}^9}{r^9}\dots.$$

Since $\dfrac{k^3\cos\beta\cos\gamma}{\sin^3\beta} = \dfrac{\lambda}{1+\lambda}\mathrm{a}^3$, it follows that $\dfrac{k^2}{\sin^2\beta} = \left(\dfrac{\lambda}{1+\lambda}\right)^{2/3}\dfrac{\mathrm{a}^2}{(\cos\beta\cos\gamma)^{2/3}}$.

The semi-axes of the ellipsoid are given by

$$\frac{c^2}{\mathrm{a}^2} = \frac{k^2}{\mathrm{a}^2\sin^2\beta} = \left(\frac{\lambda}{1+\lambda}\right)^{2/3}\frac{1}{(\cos\beta\cos\gamma)^{2/3}}, \quad \frac{a^2}{\mathrm{a}^2} = \frac{c^2}{\mathrm{a}^2}\cos^2\gamma, \quad \frac{b^2}{\mathrm{a}^2} = \frac{c^2}{\mathrm{a}^2}\cos^2\beta.$$

But $\cos^2\gamma = 1-g^2$, $\cos^2\beta = 1-\kappa^2g^2$, and therefore

$$\frac{c^2}{\mathrm{a}^2} = \left(\frac{\lambda}{1+\lambda}\right)^{2/3}(1-g^2)^{-1/3}(1-\kappa^2g^2)^{-1/3}; \quad \frac{a^2}{\mathrm{a}^2} = \left(\frac{\lambda}{1+\lambda}\right)^{2/3}(1-g^2)^{2/3}(1-\kappa^2g^2)^{-1/3};$$

$$\frac{b^2}{\mathrm{a}^2} = \left(\frac{\lambda}{1+\lambda}\right)^{2/3}(1-g^2)^{-1/3}(1-\kappa^2g^2)^{2/3}.$$

Setting apart the factor $[\lambda/(1+\lambda)]^{2/3}$, which is common to all, these three are all expressible in the form, say

$$F = 1 + (a_0+a_1\kappa^2)g^2 + (b_0+b_1\kappa^2+b_2\kappa^4)g^4 + (c_0+c_1\kappa^2+c_2\kappa^4+c_3\kappa^6)g^6 + \dots,$$

where the a_0, a_1, b_0, b_1, &c., have different numerical values according to whichever of the three functions we are treating.

Now the above formula for κ'^2 enables us to write

$$\kappa^2 = A_0 + B_0g^2 + C_0g^4 \dots,$$

where the forms of A_0, B_0, C_0 are obvious.

Hence we have

$$F = 1 + (a_0+a_1A_0)g^2 + (b_0+a_1B_0+b_1A_0+b_2A_0^2)g^4$$
$$+ (c_0+c_1A_0+c_2A_0^2+c_3A_0^3+b_1B_0+2b_2A_0B_0+a_1C_0)g^6 \dots$$

In this way I find the following expressions for the semi-axes in series proceeding by powers of g^2 or $\sin^2\gamma$:—

$$\left.\begin{aligned}
\frac{c^2}{\mathrm{a}^2} &= \left(\frac{\lambda}{1+\lambda}\right)^{2/3}\left[1+\frac{7+\lambda}{3(4+\lambda)}g^2+\frac{524+223\lambda+14\lambda^2}{3^2.7(4+\lambda)^2}g^4 + \frac{120926+86748\lambda+19428\lambda^2+686\lambda^3}{3^4.7^2(4+\lambda)^3}g^6\ldots\right]\\
\frac{a^2}{\mathrm{a}^2} &= \left(\frac{\lambda}{1+\lambda}\right)^{2/3}\left[1-\frac{5+2\lambda}{3(4+\lambda)}g^2-\frac{64+8\lambda+7\lambda^2}{3^2.7(4+\lambda)^2}g^4 - \frac{11122+2460\lambda-1851\lambda^2+196\lambda^3}{3^4.7^2(4+\lambda)^3}g^6\ldots\right]\\
\frac{b^2}{\mathrm{a}^2} &= \left(\frac{\lambda}{1+\lambda}\right)^{2/3}\left[1-\frac{2-\lambda}{3(4+\lambda)}g^2-\frac{187+110\lambda-14\lambda^2}{3^2.7(4+\lambda)^2}g^4 - \frac{28492+36723\lambda+14160\lambda^2-686\lambda^3}{3^4.7^2(4+\lambda)^3}g^6\ldots\right]
\end{aligned}\right\} \quad (66).$$

It is easy to verify that the product of the three series is unity, as should be the case.

The next step is to substitute for g^2, g^4, g^6 ... their values in terms of a^3/r^3 and its powers. In this way I find

$$\left.\begin{aligned}
\frac{c^2}{\mathrm{a}^2} &= \left(\frac{\lambda}{1+\lambda}\right)^{2/3}\left[1+\frac{5(7+\lambda)}{6(1+\lambda)}\frac{\mathrm{a}^3}{r^3}+\frac{25(419+187\lambda+11\lambda^2)}{2^2.3^2.7(1+\lambda)^2}\frac{\mathrm{a}^6}{r^6} + \frac{125(99848+74769\lambda+16503\lambda^2+488\lambda^3)}{2^3.3^4.7^2(1+\lambda)^3}\frac{\mathrm{a}^9}{r^9}\ldots\right]\\
\frac{a^2}{\mathrm{a}^2} &= \left(\frac{\lambda}{1+\lambda}\right)^{2/3}\left[1-\frac{5(5+2\lambda)}{6(1+\lambda)}\frac{\mathrm{a}^3}{r^3}+\frac{25(11+37\lambda-\lambda^2)}{2^2.3^2.7(1+\lambda)^2}\frac{\mathrm{a}^6}{r^6} - \frac{125(23992+17895\lambda+1587\lambda^2+178\lambda^3)}{2^3.3^4.7^2(1+\lambda)^3}\frac{\mathrm{a}^9}{r^9}\ldots\right]\\
\frac{b^2}{\mathrm{a}^2} &= \left(\frac{\lambda}{1+\lambda}\right)^{2/3}\left[1-\frac{5(2-\lambda)}{6(1+\lambda)}\frac{\mathrm{a}^3}{r^3}-\frac{25(157+119\lambda-11\lambda^2)}{2^2.3^2.7(1+\lambda)^2}\frac{\mathrm{a}^6}{r^6} - \frac{125(19114+23673\lambda+11577\lambda^2-488\lambda^3)}{2^3.3^4.7^2(1+\lambda)^3}\frac{\mathrm{a}^9}{r^9}\ldots\right]
\end{aligned}\right\} \quad (67).$$

By writing $1/\lambda$ for λ we obtain the formulæ for the axes of the other ellipsoid.

The numerical coefficients increase rather rapidly so that the series are useless unless a/r is small, and accordingly this method fails to give any result for ROCHE's ellipsoid in limiting stability; it is, however, useful for the problem of figures of equilibrium, as already stated.

If we had relied on spherical harmonic analysis, we should only have obtained the terms in a^3/r^3.

In order to obtain the expression for the angular momentum, which has to be a minimum for limiting stability, we must evaluate ζ. Now, from (24) and (15), we have

$$\zeta = \frac{3}{10r^2}(2c^2-a^2-b^2+\text{ same in } A, B, C)$$
$$+\frac{3}{56r^4}[3(a^4+b^4)+8c^4-8c^2(a^2+b^2)+2a^2b^2+\text{ same in } A, B, C],$$
$$+\frac{3}{20r^4}[2(A^2a^2+B^2b^2+C^2c^2)+(A^2+B^2+C^2)(a^2+b^2+c^2)-5C^2(a^2+b^2)$$
$$-5c^2(A^2+B^2)+5c^2C^2].$$

By means of the above series for a^2, b^2, c^2, and of their analogues for A^2, B^2, C^2, I find

$$\zeta = l\frac{\mathrm{a}^5}{r^5}+m\frac{\mathrm{a}^8}{r^8}+n\frac{\mathrm{a}^{10}}{r^{10}},$$

where

$$l = \tfrac{3}{4}\cdot\frac{\lambda^{2/3}(7+\lambda)+7\lambda+1}{(1+\lambda)^{5/3}},$$

$$m = \tfrac{5}{14}\cdot\frac{\lambda^{2/3}(82+38\lambda+\lambda^2)+82\lambda^2+38\lambda+1}{(1+\lambda)^{8/3}},$$

$$n = \tfrac{225}{224}\frac{\lambda^{4/3}(33+10\lambda+\lambda^2)+33\lambda^2+10\lambda+1}{(1+\lambda)^{10/3}}+\tfrac{15}{16}\frac{\lambda^{2/3}(15+102\lambda+15\lambda^2)}{(1+\lambda)^{10/3}}.$$

Now we have to evaluate the moment of momentum given above in § 18, viz.,

$$\mu_2 = \left(\frac{\mathrm{a}}{r}\right)^{3/2}(1+\zeta)^{1/2}\left[\tfrac{1}{5}\frac{\lambda}{1+\lambda}(b^2+c^2)+\tfrac{1}{5}\frac{1}{1+\lambda}(B^2+C^2)+\frac{\lambda}{(1+\lambda)^2}r^2\right].$$

When b^2, c^2, B^2, C^2, have their values attributed to them, we find

$$\mu_2 = \frac{\lambda}{(1+\lambda)^2}\left(\frac{\mathrm{a}}{r}\right)^{3/2}(1+\zeta)^{1/2}\left[R+S\frac{\mathrm{a}^3}{r^3}+T\frac{\mathrm{a}^6}{r^6}+U\frac{\mathrm{a}^9}{r^9}\ldots+r^2\right],$$

where

$$R = \frac{2}{5\lambda}(1+\lambda^{5/3})(1+\lambda)^{1/3},$$

$$S = \frac{1}{6(1+\lambda)^{2/3}}\left[\lambda^{2/3}(5+2\lambda)+\frac{1}{\lambda}(5\lambda+2)\right],$$

$$T = \frac{5}{2.3^2.7(1+\lambda)^{5/3}}\left[\lambda^{2/3}(131+34\lambda+11\lambda^2)+\frac{1}{\lambda}(131\lambda^2+34\lambda+11)\right],$$

$$U = \frac{25}{2^2.3^4.7^2(1+\lambda)^{8/3}}\Big[\lambda^{2/3}(40367+25548\lambda+2463\lambda^2+488\lambda^3)$$
$$+\frac{1}{\lambda}(40367\lambda^3+25548\lambda^2+2463\lambda+488)\Big].$$

In § 1, where the same problem is treated for two spheres, we had l, m, n, S, T, U, all zero.

In order to find the minimum moment of momentum for a given value of λ, I compute l, m, n, R, S, T, U, and assuming several equidistant values of r compute values of $\mu_2 \times \frac{(1+\lambda)^2}{\lambda}$. When the coefficients are computed we very easily find the value of r corresponding to the minimum.

When that value of r is found, we are in a position to compute the axes of the two ellipsoids.

For values of λ less than $\frac{1}{2}$ the results found in this way would be satisfactory, and for $\lambda = \frac{1}{2}$ they are, I think, adequate. Even for the case of $\lambda = 1$ the result is not very remote from the truth, for whereas the correct result for the minimum of angular momentum is $r/a = 2{\cdot}638$, the result derived from this approximate method is $r/a = 2{\cdot}51$. But it would have been impossible to foresee that the result would be as good as it is.

Part II.—Numerical Solutions.

§ 20. Roche's *Infinitesimal Ellipsoidal Satellite in Limiting Stability.*

We require to find the form of an infinitesimal satellite (so that $\lambda = 0$) revolving in a circular orbit about a *spherical* planet. When this problem is solved we shall be able to see how far the solution will be affected when we allow the spherical planet to become oblate under the influence of a rotation of the same speed as that of the revolution of the infinitesimal satellite. This last is what I have called the modified form of Roche's problem.

The planet being spherical and λ being zero, the small terms ζ, ϵ, η vanish, so that our solution becomes rigorous.

The angular momentum of the planet's axial rotation is to be omitted, and the satellite being infinitesimal the momentum of its axial rotation is zero. Thus the moment of momentum of the system varies as the square root of the satellite's radius vector, and minimum momentum coincides with minimum radius vector.

The solution of the problem has been obtained in two ways: first by Legendre's tables of elliptic integrals, and secondly by means of the auxiliary tables given above. In the first method, I knew with fair approximation by various preliminary computations the values of κ and γ which lay near to the required solution. Now there is a certain function of κ, γ, say $f(\sin^{-1}\kappa, \gamma)$, which vanishes when the ellipsoid is a figure of equilibrium; accordingly I computed by means of Legendre's tables the following eight values of $f(\sin^{-1}\kappa, \gamma)$ for integral degrees of $\sin^{-1}\kappa$ and γ:—

$$f(77°, 57°) = +0{\cdot}0000878, \quad f(77°, 58°) = -0{\cdot}0000624$$
$$f(78°, 59°) = +0{\cdot}0000724, \quad f(78°, 60°) = -0{\cdot}0000785$$
$$f(79°, 61°) = +0{\cdot}0000562, \quad f(79°, 62°) = -0{\cdot}0000939$$
$$f(80°, 63°) = +0{\cdot}0000408, \quad f(80°, 64°) = -0{\cdot}0001046$$

(probably the last significant figure in each of these is inaccurate).

Interpolating from these we find four values satisfying $f(\sin^{-1}\gamma, \kappa) = 0$, namely:—

$$f(77°, 57°{\cdot}5846) = 0,\ f(78°, 59°{\cdot}4798) = 0,\ f(79°, 61°{\cdot}3744) = 0,\ f(80°, 63°{\cdot}2806) = 0.$$

With these solutions I find

$\sin^{-1}\kappa$	r/a
77°	2·467860
78°	2·458191
79°	2·455446
80°	2·460289

By formulæ of interpolation the minimum of r occurs when $\sin^{-1}\kappa = 78°{\cdot}8756$. Then by a second interpolation this value of κ corresponds with $\gamma = 61°{\cdot}1383$, and the minimum value of r is 2·45539. We may take then $\gamma = 61°\ 8'{\cdot}3$, $\kappa = \sin 78°\ 52'{\cdot}5$, whence $\beta = 59°\ 14'{\cdot}5$. Since $\cos\gamma = 0{\cdot}4827$, $\cos\beta = 0{\cdot}5114$, the three axes of the ellipsoid are proportional to 10000, 5114, 4827; ROCHE gave the ratios 1000, 496, 469, and the radius vector as 2·44, in place of 2·45539.

Turning now to the second solution, I solved the problem by means of the auxiliary tables in two ways, namely, for $\gamma = 60°, 61°, 62°$ and also for $\gamma = 57°, 59°, 61°, 63°$.

They led to virtually identical results, viz., that the minimum of r is 2·45521, corresponding to $\gamma = 61°\ 8'{\cdot}4$, $\sin^{-1}\kappa = 78°\ 52'{\cdot}0$.

Finally the solution for ROCHE'S limit and for the ratio of the axes of the ellipsoid in limiting stability may be taken to be as follows:—

γ	$\sin^{-1}\kappa$	$\cos\gamma$	$\cos\beta$	r/a
61° 8½′	78° 52′	0·4827	0·5114	2·4553,

with uncertainty of unity in the last place of decimals in r and of half a minute of arc in $\sin^{-1}\kappa$.

We must next consider the modified form of ROCHE'S problem, in which the large body or planet yields to centrifugal force and becomes an oblate ellipsoid of revolution. The approximate formulæ of § 19 show that when $\lambda = \infty$ or when $\lambda = 0$,

$$l = \tfrac{3}{4}, \quad m = \tfrac{5}{14}, \quad n = \tfrac{225}{224}.$$

Hence in this case

$$\zeta = \tfrac{3}{4}\left(\frac{a}{r}\right)^5 + \tfrac{5}{14}\left(\frac{a}{r}\right)^8 + \tfrac{225}{224}\left(\frac{a}{r}\right)^{10} \cdots\cdot$$

The solution of the modified problem can only differ slightly from that just found when the planet is spherical, and therefore we may compute ζ with sufficient accuracy by means of the values of a/r already found. I accordingly computed ζ for $\gamma = 60°, 61°, 62°$, and found that in each case ζ was very nearly equal to 0·0088.

Now it is proved in the footnote to § 10 that, when $\lambda = 0$ and when the planet yields to centrifugal force, $\epsilon = \eta = \zeta$; as the value of ζ is found with good approximation, it is easy to compute r for these three values of γ. I thus find that in the modified problem, minimum radius vector, and therefore limiting stability, occurs when $r = 2{\cdot}457$, $\gamma = 61° 12'$, $\kappa = \sin 78° 50'$, $\beta = 59° 17'$; the axes of the large body are determined by the approximate formulæ of § 10 to be $\frac{C}{a} = \frac{B}{a} = 1{\cdot}0304$, $\frac{A}{a} = 0{\cdot}9418$.

It appears then that the yielding of the planet to centrifugal force makes very little difference, as was to be expected.

These results are included in the table given below of results for solutions of the modified problem of ROCHE with finite values of λ.

§ 21. ROCHE'S *Ellipsoidal Satellite, of finite mass, in limiting stability, the planet being also ellipsoidal.*

This is the problem which I describe as the "modified" problem of ROCHE. It seemed unnecessary to carry out the computations for the smaller values of λ, since they are sufficiently represented by the case of the infinitesimal satellite where λ is zero. I therefore begin with the case of λ = 0·4 and pass on to λ = 0·5, 0·6, 0·7, 0·8, 0·9, 1·0.

It seems well to describe the process followed in one case as a type of all. It was, in general, possible either by extrapolation from neighbouring values of λ, or by mere guessing, to begin with some values of ζ, ε, η and their co-relative functions E, H for the larger body, which were somewhere near the truth. With these we could compute r, κ, Γ, K with fair approximation; thence values of ζ, ε, η, E, H could be calculated with close accuracy and the computation could be repeated. It was, of course, a matter of conjecture as to what initial values of γ would be found to embrace the region of minimum angular momentum.

I will now describe the process for λ = 0·4. Passing over the preliminary stages in which fairly good values were found, we begin with the following conjectural values :—

γ.	46°.	48°.	50°.	Γ.	32°.	34°.	36°.
$\sin^{-1}\kappa$	68° 24′·8	69° 32′·0	70° 41′·3	$\sin^{-1}$ K	50° 12′·0	51° 18′·8	52° 34′·9
Γ	33° 13′·9	34° 18′·8	35° 18′·5	γ	43° 50′·9	47° 24′·4	51° 34′·1
$\sin^{-1}$ K	50° 52′·0	51° 30′·1	52° 7′·5	$\sin^{-1}\kappa$	67° 15′·0	69° 11′·8	71° 37′·0
log r	0·40245	0·39594	0·39060	log r	0·41071	0·39775	0·38726,

whence I compute

ζ	0·056259	0·064863	0·074219	ζ	0·047883	0·062231	0·082010
ε	0·045435	0·048102	0·050256	E	0·140179	0·174168	0·218540
η	0·32004	0·36426	0·40838	H	0·40539	0·52564	0·69272.

By means of these and the auxiliary tables I find

$\sin^{-1}\kappa$	68° 24′·6	69° 31′·8	70° 40′·5	$\sin^{-1}$ K	50° 11′·8	51° 18′·8	52° 33′·8
log r	0·40240	0·39591	0·39046	log r	0·41068	0·39768	0·38693.

The computed values are so very close to the conjectural ones, in so far as they have been as yet computed, that we might be content, but in order to illustrate the process when the conjectures are less satisfactory, I proceed to the next stage.

By far the greater part of the discrepancy between assumed and computed values (which in some cases was considerable) arises from error in the assumed values of r. Now assuming κ and K to be correct, it is very easy to correct the results for a changed value of r.

In this case I find

corrected ζ	0·056274	0·064873	0·074271	corrected ζ	0·047890	0·062253	0·082137
,, ϵ	0·04545	0·04811	0·05030	,, E	0·14020	0·17423	0·21889
,, η	0·32012	0·36431	0·40867	,, H	0·40546	0·52583	0·69385

Recomputing

$\sin^{-1}\kappa$	unchanged			$\sin^{-1}$ K	unchanged		
corrected $\log r$	0·40240	0·39591	0·39047	corrected $\log r$	0·41069	0·39768	0·38695.

By means of these we find two formulæ of interpolation, namely :—

$$\frac{r}{a} = 2·4884 - 0·0342\left(\frac{\gamma - 48^\circ}{2}\right) + 0·0032\left(\frac{\gamma - 48^\circ}{2}\right)^2,$$

$$\frac{r}{a} = 2·4985 - 0·0685\left(\frac{\Gamma - 34^\circ}{2}\right) + 0·0075\left(\frac{\Gamma - 34^\circ}{2}\right)^2.$$

These two expressions may be equated to one another, and therefore we have the means of finding simultaneous values of γ and Γ, and thence by another formula of interpolation of κ and K. Hence I obtain

γ.	46°.	48°.	50°.	Γ.	32°.	34°.	36°.
Γ	33° 13′·4	34° 17′·9	35° 17′·4	γ	43° 47′·4	47° 25′·4	51° 34′·4
$\sin^{-1}$ K	50° 52′·2	51° 29′·5	52° 6′·3	$\sin^{-1}\kappa$	67° 12′·1	69° 12′·3	71° 35′·6.

Comparison with the initial values shows that the conjectures were very good.

It now remains to compute the moment of momentum, and as we are dealing with ROCHE's problem the rotational momentum of the larger ellipsoid is not required. It follows that the values of Γ and K are not used, and since they are only required for finding the shape of the larger ellipsoid, there was no necessity for a high degree of accuracy in them. The moment of momentum is represented by the quantity μ_1 of § 18. I find then

γ.	46°.	48°.	50°.
μ_1	0·348640	0·348300	0·348519.

By formulæ of interpolation the minimum value occurs when $\gamma = 48^\circ\ 12'·9$ and $\kappa = \sin\ 69^\circ\ 39'·1$. The corresponding values are $\frac{r}{a} = 2·4848$, $\Gamma = 34^\circ\ 25'·6$, K = sin 51° 33′·5. The last step is to compute the axes of the two ellipsoids from the values of κ, γ, K, Γ.

Of course the numbers set out above make no claim to absolute accuracy, but the results tabulated below are, I believe, substantially correct.

The unit of length employed is the radius of a sphere whose mass is equal to the mass of the whole system. If it were preferred to express the results in terms of the mean radius of the larger body, all linear results would have to be multiplied by $(1+\lambda)^{1/3}$.

We may now collect the results in a tabular form, as follows :—

SOLUTIONS for ROCHE'S Ellipsoid in Limiting Stability.

The unit of length is the radius of a sphere whose mass is equal to the sum of the masses, *i.e.*, $abc+ABC = 1$, and $\dfrac{abc}{ABC} = \lambda$.

λ.	γ.	$\sin^{-1}\kappa$.	a.	b.	c.	Γ.	$\sin^{-1}$ K.	A.	B.	C.	r.
0	61° 12′	78° 50′	0·482 ÷ ∞	0·511 ÷ ∞	1·0 ÷ ∞	—	—	0·942	1·030	1·030	2·457
0·4	48° 13′	69° 39′	0·562	0·603	0·843	34° 25′	51° 34′	0·815	0·886	0·988	2·485*
0·5	46° 40′	68° 12′	0·597	0·642	0·870	35° 59′	54° 30′	0·792	0·860	0·979	2·484
0·6	45° 5′	66° 43′	0·627	0·674	0·888	37° 14′	56° 41′	0·772	0·836	0·969	2·490
0·7	43° 38′	65° 20′	0·652	0·701	0·901	38° 9′	58° 18′	0·753	0·815	0·958	2·497
0·8	42° 26′	64° 4′	0·673	0·725	0·912	38° 57′	59° 39′	0·737	0·796	0·947	2·502
0·9	41° 25′	62° 58′	0·691	0·744	0·921	39° 40′	60° 47′	0·722	0·778	0·937	2·508
1·0	40° 15′	61° 43′	0·708	0·762	0·927	40° 15′	61° 43′	0·708	0·762	0·927	2·514

The cases $\lambda = 0·4$, $0·7$, $1·0$ are illustrated by figs. 2, 3, 4. The meaning of the dotted lines near the vertices of the smaller ellipsoid will be explained in the next section.

The distance $r-(c+C)$ is the interval between the vertices of the two ellipsoids; the following are the values, using, however, more places of decimals than are tabulated above :—

λ.	$r-(c+C)$.	λ.	$r-(c+C)$.
0	1·030	0·7	0·638
0·4	0·653	0·8	0·643
0·5	0·635	0·9	0·650
0·6	0·633	1·0	0·660

It is remarkable how very nearly constant the intervening space remains throughout a large range in the values of λ.

* The values $r = 2·485$ for $\lambda = 0·4$ and $r = 2·484$ for $\lambda = 0·5$ represent 2·4848 and 2·4844 respectively; it is probable that the last significant figure in the former is a little too large and in the latter too small, and that it might have been more correct to invert the 2·485 and 2·484 in the table. I give the result, however, of the computation.

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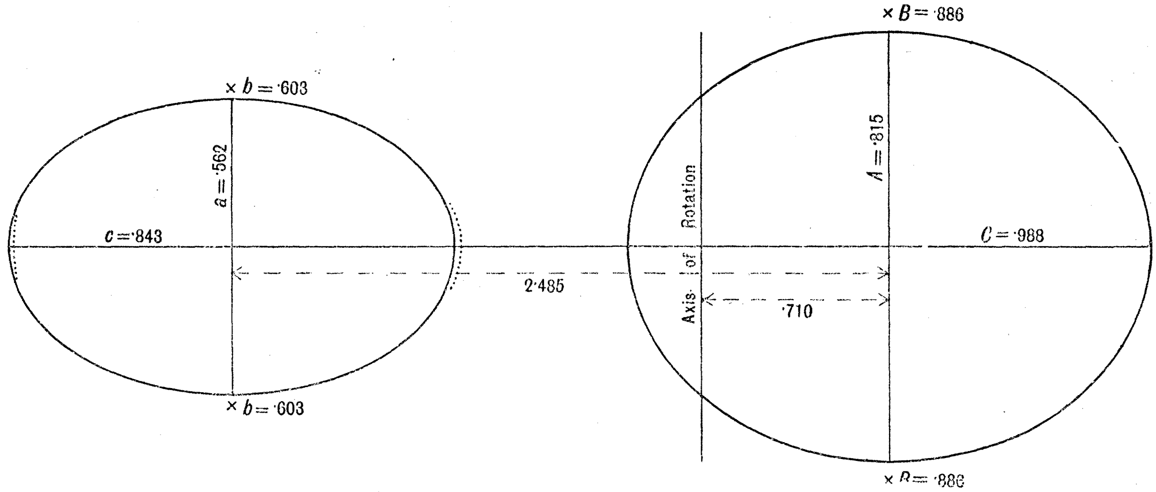

Fig. 2. ROCHE's ellipsoid in limiting stability, when $\lambda = 0\cdot4$.

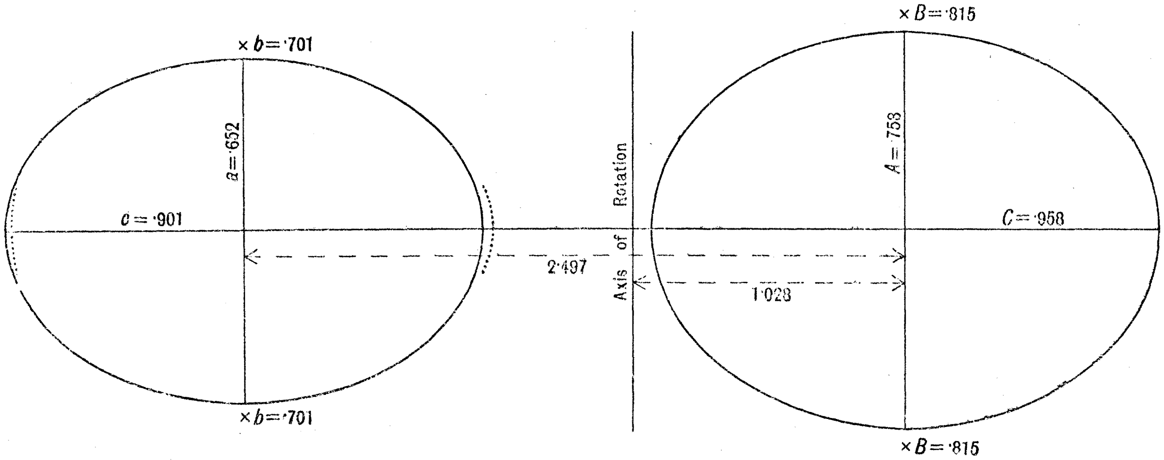

Fig. 3. ROCHE's ellipsoid in limiting stability, when $\lambda = 0\cdot7$.

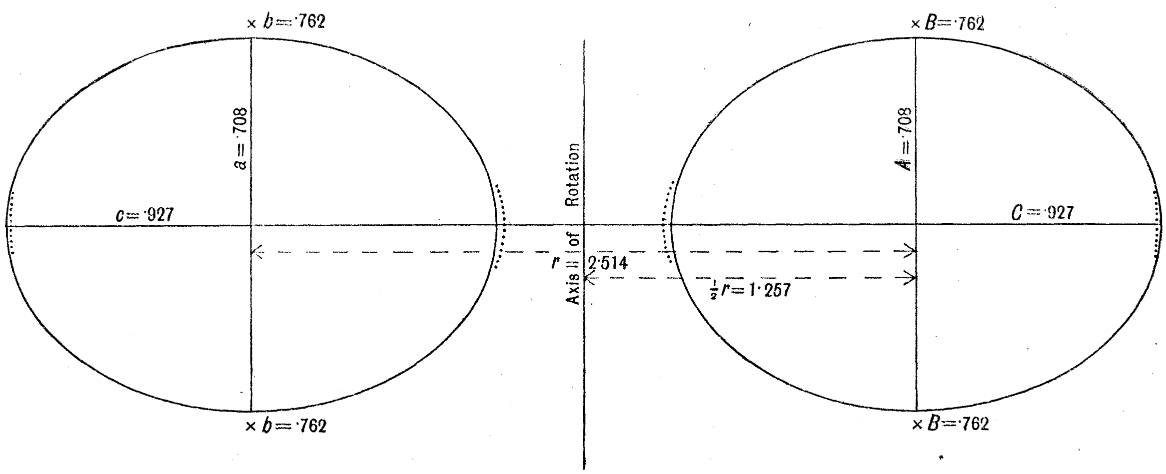

Fig. 4. ROCHE's ellipsoids in limiting stability for equal masses.

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§ 22. *Harmonic Deformations of the Ellipsoids.*

When λ is infinitely small, so that the liquid satellite is infinitely small, the harmonic deformations are evanescent, and the same is true when λ is infinitely great. We saw in § 14 that there was reason to suppose that ROCHE's ellipsoid in limiting equilibrium might be more markedly deformed for values of λ midway between zero and unity than for the latter value. I therefore determined the harmonic deformations in the three cases $\lambda = 0{\cdot}4$, $0{\cdot}7$, $1{\cdot}0$.

The formulæ for the ellipticities f_3, f_3^2, f_4, f_5 are given in §§ 14, 16, 17 and their values may be found for the ellipsoids in limiting stability tabulated in the last section. It appears that in every case the amount of deformation is small, and therefore it was sufficient to compute the normal deformation at the two extremities of the c semi-axis, that is to say, at the points nearest to and most remote from the other ellipsoid. At these points the normal displacement outwards may be denoted δc, with numbers affixed thereto so as to indicate to which harmonic it is due.

The results may be given in a tabular form, but it may be well to remark that f_3^2 (second tesseral of the third harmonics) was only computed in two instances, because its effect was obviously quite negligible. For the same reason f_4^2, f_4^4, f_5^2, f_5^4 were not computed at all. The following are the computed values of the ellipticities :—

λ.	0·4.	0·7.	1·0.
f_3	0·161	0·234	0·163
f_3^2	−0·127	−0·102	?
f_4	0·0042	0·0066	0·0059
f_5	0·0009	0·0024	0·0033

We saw reason to suppose that the higher harmonics would be relatively more important for the larger values of λ, and there is evidence of the general correctness of this view.

The values of the ellipticities afford us no idea of the amount of the normal correction, and I therefore proceed to tabulate the values of δc, the prolongations of the c semi-axis.

Towards larger ellipsoid

λ.	0·4.	0·7.	1·0.
δc_3	0·0193	0·0283	0·0194
$\delta c_3^{\,2}$	0·00039	0·00068	?
δc_4	0·00472	0·00886	0·00911
δc_5	0·00110	0·00384	0·00629
Total δc	0·0255	0·0417	0·0348
c	0·8433	0·9006	0·9270
$c+\delta c$	0·8688	0·9423	0·9618
$\dfrac{\delta c}{c}$	1/33	1/22	1/27

Away from larger ellipsoid

λ.	0·4.	0·7.	1·0.
δc_3	−0·0193	−0·0283	−0·0194
$\delta c_3{}^2$	−0·00039	−0·00068	?
δc_4	+0·00472	+0·00886	+0·00911
δc_5	−0·00110	−0·00384	−0·00629
Total δc	−0·0161	−0·0240	−0·0166
c	0·8433	0·9006	0·9270
$c+\delta c$	0·8272	0·8766	0·9104
$\frac{\delta c}{c}$	1/52	1/37	1/56

The last line in each division of this table has been given in order to show the relative importance of the total correction. It is clear that the ellipsoid remains a substantially correct solution.

These corrections to the semi-major axes are indicated by dotted lines at the extremities of the longest axis of the smaller mass in figs. 2 and 3, and of both masses in fig. 4.

We have in the last section tabulated $r-(c+C)$, the distance between the two vertices. Now, although I have not calculated the deformations of the larger ellipsoid, it is pretty clear that they must bear to those of the smaller one approximately the ratio of λ to unity. Accepting this conjecture, we have for the δC of the larger ellipsoid towards the smaller one the following values :—

λ.	0·4.	0·7.	1·0.
δC	0·010	0·029	0·035

The distance between the two surfaces of liquid is clearly $r-(c+\delta c+C+\delta C)$.

Thus we have

λ.	0·4.	0·7.	1·0.
$r-(c+C)$	0·653	0·638	0·660
$\delta c+\delta C$	0·036	0·071	0·070
$r-(c+\delta c+C+\delta C)$	0·617	0·567	0·590

§ 23. *Certain Tests and Verifications.*

In order to test how nearly the solution for a^3/r^3 by series in (40) of § 11 would agree with the solution (36) of § 10 in terms of the F elliptic integral, I computed for $\lambda = 0·7$ the value of r/a in the two ways for three values of γ, and found the following results :—

$\lambda = 0·7$.

γ.	42°.	44°.	46°.
r/a by series	2·5355	2·4888	2·4500
r/a by elliptic integrals	2·5359	2·4881	2·4495

The agreement seems to be as close as could be expected when five-figured logarithms are used.

Certain terms in ζ, as expressed in (24) of § 10, were neglected on the ground that they are of higher order than those retained. But it appears from the approximate solution in § 19 that the coefficients of the terms retained are themselves small, so that we are really only retaining terms of the same order as others which are neglected.

The most important of the neglected terms in ζ is

$$-\frac{3}{k} f_3 \mathfrak{P}_3(\nu_0)\, \mathfrak{P}_3(1)\, \mathsf{C}_3(\tfrac{1}{2}\pi)\, r^2 \frac{d}{dr} \mathfrak{Q}_3\left(\frac{r}{k}\right),$$

and this is a term of the seventh order. It seems, therefore, well to compute this in one case, and see how large a proportion it bears to the whole value.

Since

$$\mathfrak{Q}_3\left(\frac{k}{r}\right) = \frac{k^4}{7\kappa^2 r^4}\left(1+\tfrac{10}{9}\frac{k^2}{\kappa r^2}\right),$$

$$r^2 \frac{d}{dr} \mathfrak{Q}_3\left(\frac{k}{r}\right) = -\frac{4k^4}{7\kappa^2 r^3}\left(1+\tfrac{5}{3}\frac{k^2}{\kappa r^2}\right).$$

Using this value and the other approximate values given in § 14, where f_3 is determined, I find that the neglected term is

$$\frac{-\dfrac{4}{7\lambda}\dfrac{\Delta_1{}^4}{\sin^6\beta}\left(1+0\kappa'^2-\tfrac{15}{16}\kappa'^4\right)\dfrac{k^7}{r^7}\left(1+\tfrac{25}{9}\dfrac{k^2}{\kappa r^2}\right)}{\mathfrak{A}_3-\mathsf{A}_1{}^1-\dfrac{k^3}{3\lambda r^3}\dfrac{\cos^2\gamma}{\sin^2\beta}\left(1+\dfrac{3\,(3+\kappa^2)}{14\kappa^2}\dfrac{k^2}{r^2}\dots\right)}.$$

The numerical value of this, for the case of ROCHE's ellipsoid in limiting stability when $\lambda = 0{\cdot}7$, is found to be $+0{\cdot}0016$. Now, the value of ζ, as computed from the terms retained, was found to be $0{\cdot}0677$. Thus the neglected term is about one 42nd of the whole. The neglect then seems fairly justified.

I thought it worth while to discover how far the modification of ROCHE's problem, whereby the larger body is ellipsoidal, affects the result. I find that whereas it makes but little difference in the solution for any single assumed value of γ, it does make a sensible difference in the incidence of the minimum of angular momentum, and therefore of limiting stability. Thus, when $\lambda = 0{\cdot}5$, I found in one of my preliminary solutions for ROCHE's modified problem that limiting stability occurs when $r/\mathrm{a} = 2{\cdot}49$ (the more correct value is $2{\cdot}484$), but when the larger body is a sphere it occurs when $r/\mathrm{a} = 2{\cdot}35$. Thus we see that ellipticity in the larger body induces instability at a greater distance than if it were spherical. This might have been conjectured from general considerations.

§ 24. *Figures of Equilibrium of Two Masses in Limiting Stability.*

In this case both masses are liquid. We saw in § 2 that when one of the masses is infinitely small, stability only exists when the two are infinitely far apart. When λ is less than 0·5 we may obtain fair results from the approximate investigation of § 19, but for greater values of λ it is necessary to employ the laborious method adopted in determining ROCHE's ellipsoid.

When $\lambda = 0{\cdot}5$ I obtain the following approximate results for the two figures :—

$$r = 2{\cdot}574$$

$$a = 0{\cdot}62, \qquad A = 0{\cdot}81,$$
$$b = 0{\cdot}66, \qquad B = 0{\cdot}87,$$
$$c = 0{\cdot}81, \qquad C = 0{\cdot}95.$$

It is probable that the value of r derived in this way is too small.

For $\lambda = 0{\cdot}4$, I found $r = 2{\cdot}59$, but did not calculate the axes.

The only other case in which the problem has been solved is for equal masses, when $\lambda = 1$. The two ellipsoids are exactly alike, and I find limiting stability occurs for the following values :—

$$\gamma = 36^\circ\ 18', \quad \kappa = \sin 59^\circ\ 33', \quad r = 2{\cdot}638, \quad a = 0{\cdot}723, \quad b = 0{\cdot}771, \quad c = 0{\cdot}897\,;$$

and

$$r - 2c = 0{\cdot}844.$$

This is illustrated in fig. 5.

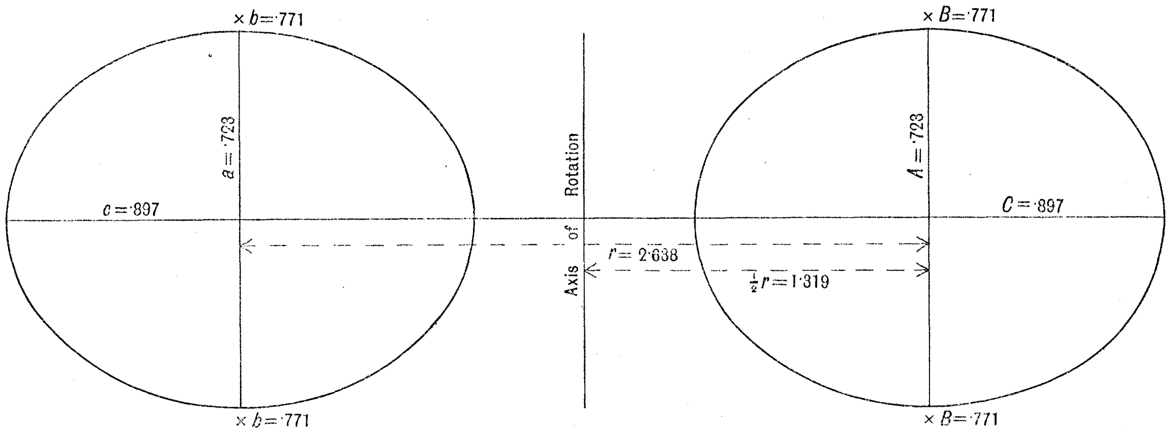

Fig. 5. Two equal masses of liquid in limiting stability.

§ 25. *Unstable Figures of Equilibrium of Two Masses.*

When $\lambda = 0$ the figure of minimum radius vector, when the larger body is rigid, is also that of minimum angular momentum, but for larger values of λ there is an ellipsoid considerably nearer to the larger body than that which possesses limiting stability. I have only determined the ellipsoid of minimum radius vector in two cases, viz., when $\lambda = 0{\cdot}8$ and $1{\cdot}0$.

When $\lambda = 0{\cdot}8$ I find minimum radius vector to be $r = 2{\cdot}36$, whereas limiting stability occurs for $r = 2{\cdot}50$. When $\lambda = 0{\cdot}8$, $r = 2{\cdot}36$, the ellipsoids are determined by the following data :—

$$\gamma = 54^\circ\ 20', \quad \kappa = \sin 71^\circ\ 51', \quad \Gamma = 46^\circ\ 10', \quad \mathrm{K} = \sin 64^\circ\ 20';$$

whence

$$a = 0{\cdot}619, \qquad A = 0{\cdot}705,$$
$$b = 0{\cdot}675, \qquad B = 0{\cdot}774,$$
$$c = 1{\cdot}063, \qquad C = 1{\cdot}018.$$

When $\lambda = 1$, the minimum radius vector occurs when γ is about 54° and is then equal to $2{\cdot}343$, whereas limiting stability occurs when $r = 2{\cdot}514$. I have not computed the axes, since it suffices to learn that there is an ellipsoidal solution when the two masses are considerably nearer than is consistent with stability.

As γ increases, the ellipsoids get longer and longer, and it is interesting to inquire whether they increase in length with such rapidity that, notwithstanding the increase of r, the interval between the two vertices continues to decrease, or whether the increase of r annuls the simultaneous increase of c.

The following table of values, computed with fair but not extreme accuracy, affords the answer to this question.

$\lambda = 1.$

γ.	r.	c.	$r-2c$.	Differences.
44°	2·429	0·962	0·506	
				−76
46°	2·396	0·983	0·430	
				−74
48°	2·370	1·007	0·356	
				−70
50°	2·354	1·034	0·286	
				−69
52°	2·345	1·064	0·217	
				−68
54°	2·343	1·097	0·149	
				−67
56°	2·350	1·134	0·082	
				−67
58°	2·367	1·176	0·015.	

The differences of $r-2c$ hardly diminish at all, and it is clear that the next entry would be negative, or in other words the two figures would overlap.

These results are obtained on the supposition that our approximation is adequate, but the small terms ζ, ϵ, η, which are really infinite series, show signs of bad convergence as γ increases. I think it probable that when we get to these extreme cases the convergence breaks down. It appears, however, justifiable to argue from these results that the unstable body continually elongates until its end coalesces with the other elongated body. I have no doubt but that the same holds true when the masses are unequal, and that we should always find $r-(c+C)$ diminishing until the two meet. The poorness of the approximation of course would prevent us from making good drawings when coalescence is approaching.

§ 26. *On the Possibility of Joining the Two Masses by a Weightless Pipe.*

This subject is considered in § 13, and it is there shown that if a certain function written $f(\mathrm{a}/r)$, for a given solution of the figures of equilibrium of two detached masses of liquid, is positive, the two masses are too far apart to admit of equilibrium when joined by a pipe without weight—and conversely.

Now I have computed $f(\mathrm{a}/r)$ in a number of cases of ROCHE'S ellipsoids in limiting stable equilibrium, and have found it always to be decisively positive.

The corresponding function for two spheres is given in (3) of § 3, and its first term is $+\mathrm{a}^3/r^3$. When we compute it for two ellipsoids, we find the corresponding term to have become negative, and the additional terms, which are given in (44), § 13, are also negative. Hence $f(\mathrm{a}/r)$ is decidedly less for two ellipsoids than it is for two spheres of the same masses with the same radius vector. Thus the deformation of the two bodies tends in the direction of making it possible to join them by a pipe without weight, but it seems certain that in the cases of the ROCHE'S ellipsoids in limiting stability such junction remains impossible.

I also computed $f(\mathrm{a}/r)$ for the much elongated ellipsoids which are roughly computed in the last section and finally overlap, and always found $f(\mathrm{a}/r)$ to be positive, as far as the approximate formula went. The additional terms tend, however, more and more to cause $f(\mathrm{a}/r)$ to vanish, and the approximation becomes very imperfect. Now I believe, although I cannot prove it rigorously, that if we could obtain a more exact evaluation of the forms of these elongated ellipsoids, and if further a more exact value of $f(\mathrm{a}/r)$ were calculable, we should find $f(\mathrm{a}/r)$ vanishing near the stage when the computations would show the two ellipsoids to overlap. It therefore seems probable that there is a figure of equilibrium consisting of two elongated masses joined by a narrow neck. These ellipsoids are very unstable when detached, and, according to the principles of § 2, it seems inconceivable that junction by a neck of fluid could render them stable.

PART III.—SUMMARY.

Since the foregoing investigation may be read by mathematicians, while astronomers and physicists will perhaps wish to learn the nature of the conclusions arrived at, I shall devote this part of the paper to a general discussion of the subject, without reference to the mathematical processes used.

Two problems are solved here simultaneously; for the analysis required for their solutions is almost identical, although the principles involved are very distinct.

We conceive that there are two detached masses of liquid in space which revolve about one another in a circular orbit without relative motion—just as the moon revolves about the earth; the determination of the shapes assumed by each mass, when in equilibrium, is common to both our problems. It is in the conditions which determine secular stability that the problem divides itself into two.

One cause of instability in the system resides in the effect on each body of the reaction on it of the frictionally resisted tides raised by it in the other. If now the larger of the two masses were rigid, while still possessing the same shape which it would have had if formed of liquid, the only effect on the orbital stability of the system would be due to the friction of the tides of the smaller mass generated by the attraction of the larger one. Investigation shows that in this case, as the two masses are brought nearer and nearer together, instability would not supervene from tidal friction until the two masses were almost in contact; but it is clear that the deformation of the figure of the liquid mass presents another possible cause of instability. In fact, instability, as due to the deformation of figure, will set in when the masses are still at a considerable distance apart. It amounts to exactly the same whether we consider the larger mass to be rigid, or whether we treat it as liquid and agree to disregard the instability which arises from the friction of the tides raised in it by the smaller body. Accordingly we may describe the stability just considered as "partial," whilst full secular stability of both bodies will depend on the tidal friction of the larger mass also.

The determination of the figure and partial stability of a liquid satellite (*i.e.*, apart from the effects of the tidal friction of the planet) is the problem of Roche. He, however, virtually regarded the planet as constrainedly a sphere, whilst in general I have treated it as an ellipsoid with the form of equilibrium.

It has already been remarked that, as the radius vector of the satellite diminishes, partial instability first supervenes from the deformation of the smaller body. It therefore hardly seems worth while to consider the partial stability of a system in which the liquid satellite (hitherto described as the smaller body) is greater than the planet. We may merely remark that in this case the problem comes to differ very little from that involved in the determination of the full secular stability of two liquid masses; for if we consider the case of a large liquid mass (the satellite) attended by a small body (the planet), it clearly makes very little difference in the result whether or not the tidal friction of the small body is included amongst the causes of instability.

This being so, I have not thought it worth while to continue the solutions of Roche's problem (modified by allowing the planet to be deformed) to the cases in which the satellite is larger than the planet. The ratio of the masses of satellite to planet is denoted above by λ, and the field examined by means of numerical solutions extends from $\lambda = 0$ to $\lambda = 1$, while the part omitted extends from $\lambda = 1$ to $\lambda = \infty$.

Tidal friction is a slowly acting cause of instability, and from the point of view of cosmical evolution the partial stability of ROCHE's ellipsoids is of even greater interest than the full secular stability of the system.

The limiting stability of ROCHE's liquid satellite is determined by the consideration that the angular momentum of the system, exclusive of the rotational momentum of the planet, shall be a minimum. This exclusion of a portion of the momentum of the whole system corresponds with the fact that we are to disregard the tidal friction of the planet as a cause of instability. If all possible cases of the liquid satellite be arranged in order of the corresponding (partial) angular momentum of the system, it is clear that for given momentum there will in general be two forms of satellite; but when the momentum is a minimum the two series coalesce. If then we proceed in order of increasing momentum, the configuration of minimum is the starting point of two series of figures; it is a figure of bifurcation, and one of the two series has one fewer degrees of instability than the other.

One of the two series is continuous with the case of a liquid satellite revolving orbitally at an infinite distance from its planet, and this is a stable configuration. Moreover, M. SCHWARZSCHILD has shown* that the whole series of ROCHE's ellipsoids does not pass through any other form of bifurcation. Hence we conclude that of the two series which start from the configuration of minimum momentum, one is stable and the other unstable.

The unstable series of solutions is continuous with a quasi-ellipsoidal satellite, infinitely elongated along the radius vector of the orbit, and the radius vector itself is infinite. Since two portions of matter cannot occupy the same space, the infinite elongation of the satellite would be physically impossible, unless the order of infinity of the radius vector were greater than that of the longest axis of the satellite. Now it appears from the numerical results of § 25 that this is not the case, and that the satellite becomes more rapidly elongated than the radius vector increases. Hence if the solution of the problem were exact we should reach a stage at which the two masses of liquid would overlap. I shall endeavour hereafter to consider the interpretation which should be put on this result.

A series of solutions for ROCHE's ellipsoid in limiting stability is tabulated in § 21, and the table gives the radius vector and the three semi-axes of each body. The unit of length adopted is the radius of a sphere whose volume is equal to the sum of the volumes of the two masses. Three of these solutions are illustrated in figs. 2, 3, 4. The section shown is that passing through the axis of rotation and the two centres, but the places are marked which the extremities of the mean axes would reach if the section had been taken at right angles to the axis of rotation.

The table of § 21 shows that the radius vector at which instability sets in only changes from 2·457 to 2·514, whilst λ, the ratio of the mass of the satellite to that of the planet, changes from zero to unity. The distance between the vertices of the

* See reference in § 18 above.

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two ellipsoids also remains wonderfully nearly constant throughout a wide range of change in the value of λ; for when $\lambda = 0\cdot4$ it is 0·653, and when $\lambda = 1$ it is 0·660, only falling to 0·633 at its minimum.

Thus far I have been speaking of the modified problem of ROCHE in which the planet assumes the appropriate figure of equilibrium, but I have also obtained the solution of ROCHE's problem for an infinitely small satellite and a spherical planet. As stated in the Preface, the radius vector of limiting stability, which has been called "ROCHE's limit," is found to be 2·4553, and the axes of the critical ellipsoid are proportional to the numbers 10000, 5114, 4827. These may be compared with the 2·44 and 1000, 496, 469 determined by ROCHE himself. When we consider the methods which he employed, we must be struck with the closeness to accuracy to which he attained.

For the infinitely small satellite "the modification" of ROCHE's problem hardly introduces any sensible change in the results, but for satellites of finite mass stability will continue to subsist for a slightly smaller radius vector for the spherical than for the ellipsoidal planet. In other words, the ellipticity of the planet induces instability earlier than would be otherwise the case.

ROCHE did not attempt to investigate how closely his equations were capable of giving the ellipsoid most nearly representative of the truth, nor did he estimate how far the ellipsoid is an accurate solution. These points are considered above, and it was the desirability of making the investigation with a closer degree of accuracy which occasioned many of the difficulties encountered.

For the infinitely small satellite the ellipsoidal solution is exact, and with a spherical planet, but not for an ellipsoidal one, ROCHE's equations give that ellipsoid exactly. In this case, however, the change introduced by the modification of ROCHE's problem is quite unimportant.

For finite satellites ROCHE's equations require sensible modification, and the solution of the "modified" problem is different from that of the unmodified one, although not to an important extent. But the ellipsoid derived from the corrected equations is deformed by an infinite series of ellipsoidal harmonic deformations, beginning with terms of the third order. Of these, the only ones which have any sensible effect are those which may be described as zonal with respect to the satellite's radius vector.

By far the most important of these is the third zonal harmonic, whereby the satellite assumes a somewhat pear-shaped figure, being sharpened towards the stalk end of the pear pointing towards the planet, and bluntened at the other end. In consequence of this deformation the shape is slightly flattened between the stalk and the middle.

The fifth and successive odd zonal harmonics accentuate the sharpening of the stalk and the bluntening of the remote end. The fourth, sixth, and successive even harmonics also accentuate the protrusion of the stalk, but tend to fill up the deficiency at the remote end.

The general effect must be very like what results from the second approximation to the pear-shaped figure of equilibrium,* for I found that the ellipsoidal form was but slightly changed over the greater part of the periphery, whilst a protrusion occurred at one end—in this present case pointing towards the planet.

In figs. 2, 3, 4, the protrusions at one end and the bluntening at the other, as computed from the third, fourth, and fifth harmonics, are indicated by dotted lines. It appears from these figures that, at least up to the point when instability sets in, the ellipsoid remains surprisingly near to the correct solution.

For an infinitely small satellite minimum radius vector also gives minimum angular momentum, so that the closest possible satellite is also in a state of limiting stability. But this is not the case for finite satellites, and there exists an unstable ellipsoidal satellite with smaller radius vector than is consistent with stability. Thus for a satellite of four-fifths of the mass of the planet the minimum radius vector is 2·36, whilst stability ceases at a distance of 2·50. Again, for equal masses stability ceases at 2·514, whilst the possibility of an ellipsoidal solution extends to 2·343.

If we follow the forms of the more and more elongated satellites, when the radius vector has begun to increase again, we find explicitly in the case of equal masses, and with practical certainty for all ratios of masses, that the distance between the two vertices continues to diminish and finally becomes negative. At this stage the two masses overlap, a conclusion which is, of course, physically impossible. But the calculation is based on the assumed adequacy of the approximations, and it is certain that the harmonic deformations of the ellipsoids increase rapidly, so that each body puts out a protrusion towards the other. The two masses of liquid must therefore really meet before we reach the stage of overlapping ellipsoids. As far as can be seen, the approximation has become very imperfect—perhaps evanescent—before the two ellipsoids cross. It will be best to continue the discussion of the meaning of this result after we have considered the true secular stability of the two masses of liquid.

If a satellite, being a particle, revolves about a rotating planet, whose tides are subject to friction, there are, for given angular momentum, two configurations (if any) in which the planet always presents the same face to the satellite. In one of these, which is unstable, the satellite is close to the planet; in the other, which is stable, it is remote.† If the angular momentum of the system be diminished, the radius vector of the stable configuration diminishes and that of the unstable one increases until the two coalesce. For yet smaller angular momentum there is no configuration possible in which the planet shall always present the same face to the satellite. We see then that amongst all possible configurations in which the planet presents the same face to

* See 'Stability,' referred to in the Preface.

† See 'Roy. Soc. Proc.,' No. 197, 1879, or Appendix G (*b*) to vol. II. of THOMSON and TAIT'S 'Natural Philosophy.'

the satellite, that one is in limiting stability, in which the two solutions coalesce with minimum angular momentum.

A rotating liquid planet will continue to repel its satellite so long as it has any rotational momentum to transfer to the orbital momentum of the satellite. Hence an infinitesimal satellite will be repelled to infinity, and the configuration of limiting stability for an infinitesimal satellite attending a planet, which always presents the same face to it, is one with infinite radius vector.

Very nearly the same conditions hold good when both planet and satellite are subject to frictional tides. In § 2 it is proved that when each body is constrainedly spherical, the radius vector of limiting stability is infinite when the ratio of the masses is infinitely small. The radius vector decreases with great rapidity as the ratio of the masses increases, and when the masses are equal, the radius vector of limiting stability is 1·738 times the radius of a sphere whose mass is equal to the sum of the masses, or is 2·19 times the radius of either of the two spheres. Thus, when the ratio of the masses falls from zero to unity (and this embraces all possible cases), limiting stability occurs with a radius vector which falls from infinity until the two spheres are only just clear of one another.

When we pass from the case of the two spheres to that of two masses, each of which is a figure of equilibrium under the attraction of itself and its companion, and subject to centrifugal force, the calculation becomes exceedingly complicated. Since the radius vector of limiting stability in every case must be greater than that of ROCHE's ellipsoid in limiting stability, and since in the latter case instability sets in through the deformation of the smaller body, it follows that in every case of true limiting secular stability of the system, instability supervenes through tidal friction.

When the ratio of the masses is small, we have seen that limiting stability occurs when the two masses are far apart. In this case the deformations of figure are small, and could easily be computed by spherical harmonic analysis.

For finite values of the ratio of masses, when spherical harmonic analysis would fail, a fair degree of exactness in the result may be obtained from the approximate formula of § 19. There would be no serious error from this formula when the ratio of masses is less than a half, but for greater values of the ratio it seems necessary to have recourse to the laborious processes employed in determining ROCHE's ellipsoids. I thought, then, that it might suffice to compute the configuration of true secular limiting stability in the case of equal masses. It is illustrated in fig. 5, and we see that the radius vector is 2·638. We found that for a pair of equal spheres, instability only set in when the radius vector, measured in the same unit, was 1·738. Thus the deformations of the two masses forbid them to approach as near to one another as if they were spheres. It should be noted that instability in this case must arise from tidal friction, because ROCHE's ellipsoid in limiting stability was found to have a radius vector of 2·514.

When POINCARÉ announced that there is a figure of equilibrium bearing some resemblance to a pear, he also conjectured that the constriction between the stalk and the middle of the pear might become developed until it became a thin neck of liquid joining two bulbs, and that yet further the neck might break and the two masses become detached. References to my own papers on this pear-shaped figure and its stability are given above in the preface, and the present investigation was undertaken in the hope that a revision of ROCHE's work would throw some light on the figure when the constriction has developed into a thin neck of liquid.

As a preliminary to greater exactness, I have in § 3 considered the motion of two masses of liquid, each constrainedly spherical, and joined to one another by a weightless pipe. Through such a pipe liquid can pass from one sphere to the other, and it will continue to do so until, for given radius vector, the spheres bear some definite ratio to one another; or, to state the matter otherwise, two spherical masses of given ratio, revolving in a circular orbit without relative motion, can be started with some definite radius vector so that liquid will not flow from one to the other.

In this system the ratio of the masses and the radius vector are the only parameters, and I find that the condition of equilibrium is a cubic equation in the radius vector with coefficients which are functions of the ratio of the masses. The cubic has three real roots of which only one has a physical meaning, and the solution is illustrated graphically in fig. 1. The single circle on the right is the larger sphere, and it is maintained of constant size for convenience of illustration. The smaller circles on the left represent the solutions for various ratios of masses, which are the cubes of the numbers written on the successive circles.

The solution of this problem seems to me very curious, but it does not possess much physical interest, since it is proved in § 3 that all the solutions are unstable.

The distance between the two masses is much smaller than is the case with any of ROCHE's ellipsoids, even with minimum radius vector, and accordingly it did not seem probable that the parallel problem, when the two masses are liquid and deformed, would possess any solution at all; nevertheless, it was worth while to pursue the investigation to the end.

When the masses are ellipsoidal and are joined by a weightless pipe, the solution would become very complicated, but the question may be attacked indirectly. When the masses are spherical there is a certain function of the radius vector and of the ratio of the masses which must vanish when a channel of communication is opened between them. If this function be computed for two given spherical masses with given radius vector, we find that it is negative if the two masses are too close together to admit of junction by a pipe without disturbance of their relative masses, and that it is positive if they are too far apart.

When the figures of equilibrium of two detached masses of liquid are determined, it is possible to form the corresponding function, but part of it consists of an infinite series of which it is only practically possible to give the first few terms. Now I have computed this function in a number of cases of ROCHE's ellipsoids, and have found that the few terms of the infinite series are small, that the series is apparently rapidly convergent, and that the function is decisively positive. We may conclude then that in none of the cases, for which numerical results have been given, is it even approximately possible to make a junction between the masses; and even if we could do so, the system would be unstable, because removal of a constraint may destroy but cannot impart stability. To find any possible solution we must consider cases where the two masses are much closer together.

I think, however, that there must be a figure of the kind sought, for the following reasons: If the function referred to above be formed for given radius vector and ratio of masses, we find that its value is very much less than if the two masses are spherical. Thus the tendency of liquid to flow from the larger to the smaller mass (when they are too far apart) is much less than if the two masses were spherical. Every increase of ellipticity in the ellipsoids tends to diminish the function, and the series tends to become less convergent; and besides I have made no attempt to evaluate the terms in the function which correspond to the harmonic inequalities of the ellipsoids, and these would tend to diminish the function still further.

It was remarked above that two much elongated ellipsoids seem to coalesce finally, but that the approximations were not satisfactory. I find, however, that even to the end the function, as far as it could be computed, was still positive although much diminished. It appears to me then probable that if we could obtain a more complete expression for the function, we should find that it vanishes before the two ellipsoids overlap. There is then some reason to believe in the existence of a figure of equilibrium consisting of two quasi-ellipsoids joined by a narrow neck; but such a figure must be unstable.

I have, in fig. 6, made a highly conjectural drawing of such a figure where the two

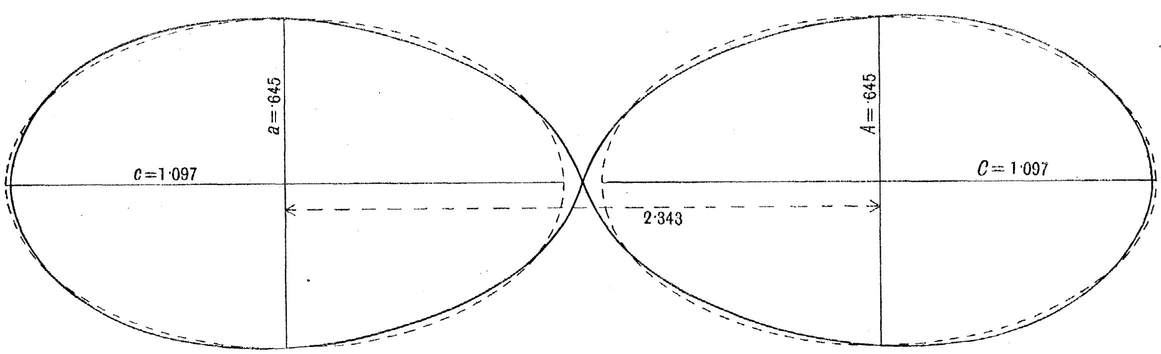

Fig. 6. Conjectural drawing of unstable figure of two equal masses of liquid just in contact.

bulbs are equal. The data are derived from the computations for the much elongated ellipsoids just before they are found to overlap.

Mr. JEANS has considered the equilibrium and stability of infinite rotating cylinders of liquid. This is the two-dimensional analogue of the three-dimensional problem.* He finds solutions perfectly analogous to MACLAURIN'S and JACOBI'S ellipsoids and to the pear-shaped figure. In consequence of the greater simplicity of the conditions, he is able to follow the development of the cylinder of pear-shaped section until the neck joining the two parts has become quite thin. His analysis, besides, points to the rupture of the neck, although the method fails to afford the actual shapes and dimensions in this last stage of development.

He is able to prove conclusively that the cylinder of pear-shaped section is stable, and it is important in connection with our present investigation to note that he finds no evidence of any break in the stability of that cylinder up to its division into two parts.

The stability of MACLAURIN'S and of the shorter Jacobian ellipsoids is, of course, well established, and I imagined that the pear-shaped figure with incipient furrowing was also proved to be stable. But M. LIAPOUNOFF now states† that he is able to prove the pear-shaped figure to be unstable from the beginning, and he attributes the discrepancy between our conclusions to the fact that my result depended on the supposed rapid convergency of an infinite series, of which only a few terms were computed. The terms computed diminish rapidly, and it seemed to me evident that the rapid diminution must continue, so that I feel unable to accept the hypothesis that the sum of the neglected terms could possibly amount to the very considerable total which would be necessary to reverse my conclusion. I am, therefore, still of opinion that the pear-shaped figure is stable at the beginning; and this view receives a powerful confirmation from MR. JEANS'S researches. The final decision must await the publication of M. LIAPOUNOFF'S investigation.

But there is another difficulty raised by the present paper. I had fully expected to find an approximation to a stable figure consisting of two bulbs joined by a thin neck, but while my work indicates the existence of such a figure, it seems to me, at present, conclusive against its stability. The weightless pipe joining two bulbs of fluid is clearly only a crude representative of a neck of fluid, but I find it hard to imagine that it is so very imperfect that the reality should be stable, while the representation is unstable. My present investigation shows that two quasi-ellipsoids just detached from one another do not possess secular stability. The vertices of such bodies would be blunt points nearly in contact; the introduction of a short pipe without weight between these blunt points would differ exceedingly little from two sharp points actually in contact. Is it possible that the difference would produce all the change from great instability even to limiting stability? The opening of a channel between the two masses is the removal of a constraint; the system does not possess true secular stability when the channel is closed, and we should have to believe that the removal of a constraint induces stability; and this is, I think, impossible.

If, then, MR. JEANS is right in believing in the stable transition from the single

* "On the Equilibrium of Rotating Liquid Cylinders," 'Phil. Trans.,' A, vol. 200, pp. 67–104.

† See reference in Preface.

cylinder to two revolving about one another, and if I am correct now, the two problems must part company at some undetermined stage. M. LIAPOUNOFF will no doubt contend that it is at the beginning of the pear-shaped series, but for the present I should disagree with such an opinion.

I have no suggestion to make as to the stage at which the pear-shaped figure may become unstable, or as to the figure which must be coalescent with it when instability supervenes. These points must await the elucidation which they will no doubt receive from future investigations.

One question remains: If my present conclusions are correct, do they entirely destroy the applicability of this group of ideas to the explanation of the birth of satellites or of double stars? I think not, for we see how a tendency to fission arises, and it is not impossible that a period of turbulence may naturally supervene in the process of separation. Finally, as Mr. JEANS points out, heterogeneity of density introduces new and important differences in the conditions.